Tightness of the weight-distribution bound for some strongly regular graphs

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Eigenfunctions of graphs

Let $\Gamma = (V, E)$ be a k-regular graph on n vertices and θ be an eigenvalue of its adjacency matrix A. Let $u = (u_1, \ldots, u_n)^t$ be an eigenvector of A corresponding to θ. Then u defines a function $f_u : V \mapsto \mathbb{R}$, which is called a θ-eigenfunction of Γ.

For an eigenfunction f_u of Γ, the *support* is the set

$$Supp(f_u) := \{x \in V \mid f_u(x) \neq 0\}.$$

MS-problem

The following problem was first formulated in [1] (see also [2] for the motivation and details).

Problem 1 (MS-problem)
Given a graph Γ and its eigenvalue θ, find the minimum cardinality of the support of a θ-eigenfunction of Γ.

A θ-eigenfunction having the minimum cardinality of support is called optimal.

Problem 2
Given a graph Γ and its eigenvalue θ, characterise optimal θ-eigenfunctions of Γ.

[1] K. V. Vorobev, D. S. Krotov, *Bounds for the size of a minimal 1-perfect bitrade in a Hamming graph*, Journal of Applied and Industrial Mathematics 9(1) (2015) 141–146, translated from Discrete Analysis and Operations Research 21(6) (2014) 3–10.

[2] E. Sotnikova, A. Valyuzhenich, *Minimum supports of eigenfunctions of graphs: a survey*, https://arxiv.org/abs/2102.11142

A survey on Problem 2

Recently, Problem 2 was solved for several classes of graphs:

- all eigenvalues of Hamming graphs $H(n, q)$ when $q = 2$ or $q > 4$ and some eigenvalues of $H(n, q)$ when $q = 3, 4$;
- all eigenvalues of Johnson graphs (asymptotically);
- the smallest eigenvalue of Hamming, Johnson and Grassmann graphs;
- the largest non-principal eigenvalue of a Star graph S_n, $n \geq 8$;
- the largest non-principal eigenvalue of Doob graphs.

A survey on Problem 1

Excepting the results from the previous slide, Problem 1 was solved for several more classes of graphs:

- both non-principal eigenvalues of Paley graphs of square order;
- strongly regular bilinear forms graphs over a prime field.

Weight-distribution bound

Let Γ be a distance-regular graph of diameter $D(\Gamma)$ with intersection array $(b_0, b_1, \ldots, b_{D(\Gamma)-1}; c_1, c_2, \ldots, c_{D(\Gamma)})$.

For an eigenvalue θ of Γ, the following bound was proposed in [3, Corollary 1].

Theorem (Weight-distribution bound)

A θ-eigenfunction f of Γ has at least $\sum_{i=0}^{D(G)} |W_i|$ nonzeros, where

$$W_0 = 1,$$

$$W_1 = \theta$$

and

$$W_i = \frac{(\theta - a_{i-1})W_{i-1} - b_{i-2}W_{i-2}}{c_i}.$$

[3] D. S. Krotov, I. Yu. Mogilnykh, V. N. Potapov, *To the theory of q-ary Steiner and other-type trades*, Discrete Mathematics 339 (3) (2016) 1150–1157.

Known results when the weight-distribution bound is tight

- the eigenvalue -1 of the Boolean Hamming graph of an odd dimension and the minimum eigenvalue of an arbitrary Hamming graph;
- both non-principal eigenvalues of Paley graphs of square order;
- the minimum eigenvalue of Johnson graphs;
- the minimum eigenvalue of Grassmann graphs;
- the minimum eigenvalue of strongly regular bilinear forms graphs over a prime field.

Tightness of the weight-distribution bound for the smallest eigenvalue of a DRG

It was shown in [3] that, for the smallest eigenvalue of a distance-regular graph Γ, the tightness of the weight-distribution bound is equivalent to the existence of an isometric bipartite distance-regular induced subgraph $T_0 \cup T_1$, where T_0 and T_1 are parts, such that an optimal eigenfunction, up to multiplication by a non-zero constant, has the following form:

$$f(x) = \begin{cases} 1, & \text{if } x \in T_0; \\ -1, & \text{if } x \in T_1; \\ 0, & \text{otherwise.} \end{cases}$$

[3] D. S. Krotov, I. Yu. Mogilnykh, V. N. Potapov, *To the theory of q-ary Steiner and other-type trades*, Discrete Mathematics 339 (3) (2016) 1150–1157.

Tightness of the weight-distribution bound for a non-principal eigenvalue of an SRG

If Γ is a strongly regular graph with non-principal eigenvalues r, s, where $s < 0 < r$, the following holds.

Lemma 1 ([3], Weight-distribution bound for SRG)
(1) An s-eigenfunction f of Γ has at least $(-2s)$ nonzeros; $|Supp(f)|$ meets the bound if and only if there exists an induced complete bipartite subgraph with parts T_0, T_1 of size $-s$;
(2) An r-eigenfunction f of Γ has at least $2(r+1)$ nonzeros; $|Supp(f)|$ meets the bound if and only if there exists an induced disjoint union of two cliques T_0, T_1 of size $r+1$.

[3] D. S. Krotov, I. Yu. Mogilnykh, V. N. Potapov, *To the theory of q-ary Steiner and other-type trades*, Discrete Mathematics 339 (3) (2016) 1150–1157.

Tightness of the weight-distribution bound for Paley graphs of square order

In [4], for Paley graphs $P(q^2)$, we showed the tightness of the weight-distribution bound for both non-principal eigenvalues, which are $s = \frac{-1-q}{2}$ and $r = \frac{-1+q}{2}$.

Let β be a primitive element in $\mathbb{F}_{q^2}$. Put $\omega := \beta^{q-1}$. Then $Q = \langle \omega \rangle$ is the subgroup of order $q+1$ in $\mathbb{F}^*_{q^2}$.

Facts about Q:

- Q is an oval in the corresponding affine plane;
- Q is the kernel of the norm mapping $N : \mathbb{F}^*_{q^2} \mapsto \mathbb{F}^*_q$, which means that $Q = \{\gamma \in \mathbb{F}^*_{q^2} \mid \gamma^{q+1} = 1\}$, or, equivalently, $Q = \{x + y\alpha \mid x, y \in \mathbb{F}_q, x^2 - y^2 d = 1\}$, where d is a non-square in $\mathbb{F}^*_q$ and $\alpha^2 = d$.

[4] S. Goryainov, V. Kabanov, L. Shalaginov, A. Valyuzhenich, *On eigenfunctions and maximal cliques of Paley graphs of square order*, Finite Fields and Their Applications 52 (2018) 361–369.

Tightness of the weight-distribution bound for Paley graphs of square order

Let $Q_0 = \langle \omega^2 \rangle$ and $Q_1 = \omega Q_0$.

Facts about Q:

- if $q \equiv 1(4)$, then $Q = Q_0 \cup Q_1$ induces a complete bipartite graph with parts Q_0 and Q_1;
- if $q \equiv 3(4)$, then $Q = Q_0 \cup Q_1$ induces a pair of disjoint cliques Q_0 and Q_1.

Corollary 1

The weight-distribution bound is tight for both non-principal eigenvalues of Paley graphs of square order.

Knowing the structure of Q, we were also able to construct new maximal cliques of the second largest known size in Paley graphs of square order (see [4]).

[4] S. Goryainov, V. Kabanov, L. Shalaginov, A. Valyuzhenich, *On eigenfunctions and maximal cliques of Paley graphs of square order*, Finite Fields and Their Applications 52 (2018) 361–369.

Generalised Paley graphs of square order; WDB for the smallest eigenvalue

Let $m > 1$ be a positive integer. Let q be an odd prime power, $q \equiv 1 \ (2m)$. The m-Paley graph on $\mathbb{F}_q$, denoted $GP(q, m)$, is the Cayley graph $Cay(\mathbb{F}_q^+, (\mathbb{F}_q^*)^m)$, where $(\mathbb{F}_q^*)^m$ is the set of m-th powers in $\mathbb{F}_q^*$.

We consider the graphs $GP(q^2, m)$, where q is an odd prime power and m divides $q + 1$; these graphs are strongly regular and form a generalisation of Paley graphs of square order (the usual Paley graphs of square order are just 2-Paley graphs of square order).

The eigenvalues of $GP(q^2, m)$ are $s = (-\frac{q+1}{m})$ and $r = \frac{(m-1)q-1}{m}$.

Given an odd prime power q and an integer $m > 1$ such that m divides $q + 1$, a $(-\frac{q+1}{m})$-eigenfunction of the generalised Paley graph $GP(q^2, m)$ has at least $\frac{2(q+1)}{m}$ non-zeroes.

Structure of Q (I)

Let us divide Q into m parts

$$Q = Q_0 \cup Q_1 \cup \ldots \cup Q_{m-1},$$

where $Q_0 = \langle \omega^m \rangle$, $Q_1 = \omega Q_0$, $\ldots$, $Q_{m-1} = \omega^{m-1} Q_0$.

Lemma 2 (G., Shalaginov, 2021+)
Let q be a prime power and m be an integer such that $m > 1$, m divides $q + 1$. The mapping $\gamma \mapsto \gamma^{q-1}$ is a homomorphism from $\mathbb{F}^*_{q^2}$ to Q. Moreover, an element γ is an m-th power in $\mathbb{F}^*_{q^2}$ if and only if γ^{q-1} is an m-th power in Q.

Lemma 3 (G., Shalaginov, 2021+)
Let γ be an arbitrary element from Q, $\gamma \neq 1$. Then, for the image of $(\gamma - 1)$ under the action of the homomorphism, the following equality holds:

$$(\gamma - 1)^{q-1} = -\frac{1}{\gamma}.$$

Structure of Q (II)

The following theorem basically states that each of the sets $Q_0, Q_1 \ldots, Q_{m-1}$ induces either a clique or an independent set, and there are at most two cliques among them.

Moreover, the theorem states that for every independent set Q_{i_1}, there exists uniquely determined independent set Q_{i_2} among $Q_0, Q_1 \ldots, Q_{m-1}$ such that there are all possible edges between Q_{i_1} and Q_{i_2} and there are no edges between Q_{i_1} and $Q \setminus Q_{i_2}$.

Structure of Q (III)

Theorem 1 (G., Shalaginov, 2021+)
Given an odd prime power q and an integer $m > 1$, m divides $q+1$, the following statements hold for the subgraph of $GP(q^2, m)$ induced by Q.
(1) If m divides $\frac{q+1}{2}$ and m is odd, then Q_0 is a clique, and $Q_1, \ldots, Q_{m-1}$ are independent sets; moreover, for any distinct i_1, i_2 such that $0 \leq i_1 < i_2 \leq m-1$, there are all possible edges between the sets Q_{i_1} and Q_{i_2} if

$$i_1 + i_2 \equiv 0 \ (mod\ m),$$

and there are no such edges if

$$i_1 + i_2 \not\equiv 0 \ (mod\ m).$$

In particular, $Q_1 \cup Q_{m-1}, \ldots, Q_{\frac{m-1}{2}} \cup Q_{\frac{m+1}{2}}$ induce $\frac{m-1}{2}$ complete bipartite graphs.

Structure of Q (IV)

(2) If m divides $\frac{q+1}{2}$ and m is even, then $Q_0, Q_{\frac{m}{2}}$ are cliques, and $Q_1, \ldots, Q_{\frac{m}{2}-1}, Q_{\frac{m}{2}+1}, \ldots, Q_{m-1}$ are independent sets; moreover, for any distinct i_1, i_2 such that $0 \leq i_1 < i_2 \leq m-1$, there are all possible edges between the sets Q_{i_1} and Q_{i_2} if

$$i_1 + i_2 \equiv 0 \ (mod\ m) \text{ and } \{i_1, i_2\} \neq \{0, \frac{m}{2}\}$$

and there are no such edges if

$$i_1 + i_2 \not\equiv 0 \ (mod\ m) \text{ or } \{i_1, i_2\} = \{0, \frac{m}{2}\}.$$

In particular, $Q_1 \cup Q_{m-1}, \ldots, Q_{\frac{m}{2}-1} \cup Q_{\frac{m}{2}+1}$ induce $(\frac{m}{2} - 1)$ complete bipartite graphs.

Structure of Q (V)

(3) If m does not divide $\frac{q+1}{2}$, then m is even.
(3.1) If $\frac{m}{2}$ is odd, then $Q_0, Q_1, \ldots, Q_{m-1}$ are independent sets; moreover, for any distinct i_1, i_2 such that $0 \leq i_1 < i_2 \leq m-1$, there are all possible edges between the sets Q_{i_1} and Q_{i_2} if

$$i_1 + i_2 \equiv \frac{m}{2} \ (mod\ m),$$

and there are no such edges if

$$i_1 + i_2 \not\equiv \frac{m}{2} \ (mod\ m).$$

In particular, if $m = 2$, $Q = Q_0 \cup Q_1$ is a complete bipartite graph; if $m \geq 6$,
$Q_0 \cup Q_{\frac{m}{2}}, \ldots, Q_{\frac{m-2}{4}} \cup Q_{\frac{m+2}{4}}, Q_{\frac{m}{2}+1} \cup Q_{m-1}, \ldots, Q_{\frac{3m-2}{4}} \cup Q_{\frac{3m+2}{4}}$
induce $\frac{m}{2}$ complete bipartite graphs.

Structure of Q (VI)

(3.2) If $\frac{m}{2}$ is even, then $Q_{\frac{m}{4}}, Q_{\frac{3m}{4}}$ are cliques, and $Q_0, \dots, Q_{\frac{m}{4}-1}, Q_{\frac{m}{4}+1}, \dots, Q_{\frac{3m}{4}-1}, Q_{\frac{3m}{4}+1}, \dots, Q_{m-1}$ are independent sets; moreover, for any distinct i_1, i_2 such that $0 \leq i_1 < i_2 \leq m-1$, there are all possible edges between the sets Q_{i_1} and Q_{i_2} if

$$i_1 + i_2 \equiv \frac{m}{2} \ (mod\ m) \text{ and } \{i_1, i_2\} \neq \{\frac{m}{2}, \frac{3m}{2}\},$$

and there are no such edges if

$$i_1 + i_2 \not\equiv \frac{m}{2} \ (mod\ m) \text{ or } \{i_1, i_2\} = \{\frac{m}{2}, \frac{3m}{2}\}.$$

In particular, if $m = 4$, $Q_0 \cup Q_2$ is a complete bipartite graph; if $m \geq 8$, then $Q_0 \cup Q_{\frac{m}{2}}, \dots, Q_{\frac{m-4}{4}} \cup Q_{\frac{m+4}{4}}, Q_{\frac{m}{2}+1} \cup Q_{m-1}, \dots, Q_{\frac{3m-4}{4}} \cup Q_{\frac{3m+4}{4}}$ induce $\frac{m-2}{2}$ complete bipartite graphs.

Structure of Q (VII) and tightness of WDB for the smallest eigenvalue of $GP(q^2, m)$

Corollary 2

Let q be an odd prime power and m be an integer $m \geq 2$, m divides $q+1$. Then, except for the case $m = 2$ and 2 divides $\frac{q+1}{2}$, there is at least one pair Q_{i_1}, Q_{i_2} among $Q_0, \ldots, Q_{m-1}$ such that $Q_{i_1} \cup Q_{i_2}$ induces a complete bipartite subgraph.

Corollary 3

Let q be an odd prime power and m be an integer $m \geq 2$, m divides $q+1$. Then the weight-distribution bound is tight for the eigenvalue $(-\frac{q+1}{m})$ of $GP(q^2, m)$.

Strongly regular graphs related to polar spaces

- Affine polar graphs $VO^+(2e, q)$
- Affine polar graphs $VO^-(2e, q)$
- Orthogonal graphs $O(2e+1, q)$, $O^+(2e, q)$ and $O^-(2e, q)$
- Symplectic graphs $Sp(2e, q)$
- Unitary graphs $U(n, q)$

For each of these families of strongly regular graphs, we show the tightness of the weight-distribution bound for the positive non-principal eigenvalue r by constructing a pair of induced isolated cliques of size $r+1$.

[5] A. E. Brouwer, Affine polar graphs,
`https://www.win.tue.nl/~aeb/graphs/VO.html`

[6] A. E. Brouwer, Families of graphs,
`https://www.win.tue.nl/~aeb/graphs/srghub.html`

[7] A. E. Brouwer, Symplectic graphs,
`https://www.win.tue.nl/~aeb/graphs/Sp.html`

[8] A. E. Brouwer, W. H. Haemers, *Spectra of Graphs*, Springer-Verlag, New York (2012).

Hyperbolic quadric

Let V be a $(2e)$-dimensional vector space over a finite field $\mathbb{F}_q$, where $e \geq 2$ and q is a prime power, provided with the hyperbolic quadratic form

$$HQ(x) = x_1x_2 + x_3x_4 + \ldots + x_{2e-1}x_{2e}.$$

The set Q^+ of zeroes of HQ is called the hyperbolic quadric, where e is the maximal dimension of a subspace in Q^+. A generator of Q^+ is a subspace of maximal dimension e in Q^+.

Lemma 4 ([9, Theorem 7.130])
Given an $(e-1)$-dimensional subspace W of Q^+, there are precisely two generators that contain W.

[9] B. De Bruyn, *An Introduction to Incidence Geometry*, Frontiers in Mathematics, Birkhäuser Basel (2016).

Affine polar graphs $VO^+(2e, q)$ (I)

Denote by $VO^+(2e, q)$ the graph on V with two vectors x, y being adjacent if and only if $Q(x - y) = 0$. The graph $VO^+(2e, q)$ is known as an affine polar graph.

Lemma 5
The graph $VO^+(2e, q)$ is a vertex-transitive strongly regular graph with parameters

$$\begin{aligned} v &= q^{2e} \\ k &= (q^{e-1} + 1)(q^e - 1) \\ \lambda &= q(q^{e-2} + 1)(q^{e-1} - 1) + q - 2 \\ \mu &= q^{e-1}(q^{e-1} + 1) \end{aligned} \tag{1}$$

and eigenvalues $r = q^e - q^{e-1} - 1$, $s = -q^{e-1} - 1$.

Affine polar graphs $VO^+(2e, q)$ (II)

Note that $VO^+(2e, q)$ is isomorphic to the graph defined on the set of all $(2 \times e)$-matrices over $\mathbb{F}_q$ of the form

$$\begin{pmatrix} x_1 & x_3 & \dots & x_{2e-1} \\ x_2 & x_4 & \dots & x_{2e} \end{pmatrix},$$

where two matrices are adjacent if and only if the scalar product of the first and the second rows of their difference is equal to 0.

Lemma 6
There is a one-to-one correspondence between cosets of generators of Q^+ and maximal cliques in $VO^+(2e, q)$.

Lemma 7
Every maximal clique in $VO^+(2e, q)$ is a q^{e-1}-regular q^e-clique.

An optimal $(q^e - q^{e-1} - 1)$-eigenfunction of $VO^+(2e, q)$

In view of Lemmas 1 and 5, a $(q^e - q^{e-1} - 1)$-eigenfunction of $VO^+(2e, q)$ whose cardinality of support meets the weight-distribution bound is given by a pair of disjoint cliques of size $(q^e - q^{e-1})$, and the cardinality of support is $2(q^e - q^{e-1})$. Take the $(e-1)$-dimensional subspace

$$W = \begin{pmatrix} * & \dots & * & 0 \\ 0 & \dots & 0 & 0 \end{pmatrix},$$

where the size of matrices is $2 \times e$. According to Lemma 4, the subspace W is contained in exactly two generators: these are

$$W_0 = \begin{pmatrix} * & \dots & * & * \\ 0 & \dots & 0 & 0 \end{pmatrix} \text{ and } W_1 = \begin{pmatrix} * & \dots & * & 0 \\ 0 & \dots & 0 & * \end{pmatrix}.$$

The cliques W_0 and W_1 are q^{e-1}-regular and have q^{e-1} vertices in common. Thus, the sets $W_0 \setminus W$ and $W_1 \setminus W$ induce a pair of disjoint cliques of size $(q^e - q^{e-1})$, which means that the weight-distribution bound is tight for the eigenvalue $(q^e - q^{e-1} - 1)$.

Elliptic quadric

Let V be a $(2e)$-dimensional vector space over a finite field $\mathbb{F}_q$, where $e \geq 2$ and q is a prime power, provided with the elliptic quadratic form

$$EQ(x) = p(x_1, x_2) + x_3x_4 + \ldots + x_{2e-1}x_{2e},$$

where $p(x_1, x_2)$ is an irreducible homogeneous polynomial of degree 2 (it means that $p(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2$, $a \neq 0$, $c \neq 0$).

The set Q^- of zeroes of EQ is called the elliptic quadric, where $e - 1$ is the maximal dimension of a subspace in Q^-. A generator of Q^- is a subspace of maximal dimension $e - 1$ in Q^- (see [9]).

[9] B. De Bruyn, *An Introduction to Incidence Geometry*, Frontiers in Mathematics, Birkhäuser Basel (2016).

Affine polar graphs $VO^-(2e, q)$ (I)

Denote by $VO^-(2e, q)$ the graph on V with two vectors x, y being adjacent if and only if $Q(x - y) = 0$. The graph $VO^-(2e, q)$ is known as an affine polar graph.

Lemma 8

The graph $VO^-(2e, q)$ is a vertex-transitive strongly regular graph with parameters

$$\begin{aligned} v &= q^{2e} \\ k &= (q^{e-1} - 1)(q^e + 1) \\ \lambda &= q(q^{e-2} - 1)(q^{e-1} + 1) + q - 2 \\ \mu &= q^{e-1}(q^{e-1} - 1) \end{aligned} \tag{2}$$

and eigenvalues $r = q^{e-1} - 1$, $s = -q^e + q^{e-1} - 1$.

Affine polar graphs $VO^-(2e, q)$ (II)

Note that $VO^-(2e, q)$ is isomorphic to the graph defined on the set of all $(2 \times e)$-matrices over $\mathbb{F}_q$ of the form

$$\begin{pmatrix} x_1 & x_3 & \dots & x_{2e-1} \\ x_2 & x_4 & \dots & x_{2e} \end{pmatrix}, \qquad (3)$$

where two matrices are adjacent if and only if the modified scalar product (for the first column we take $p(x_1, x_2)$ instead of $x_1 x_2$) of the first and the second rows of their difference is equal to 0.

An optimal $(q^{e-1}-1)$-eigenfunction of $VO^-(2e,q)$

In view of Lemmas 1 and 8, a $(q^{e-1}-1)$-eigenfunction of $VO^-(2e,q)$ whose cardinality of support meets the weight-distribution bound is given by a pair of disjoint cliques of size q^{e-1}, and the cardinality of support is $2q^{e-1}$.
Consider the generator

$$U = \begin{pmatrix} 0 & * & \dots & * \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

and its additive shift

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{pmatrix} + U = \begin{pmatrix} 1 & * & \dots & * \\ 0 & 0 & \dots & 0 \end{pmatrix},$$

which are cliques of size q^{e-1}. It is easy to see that there are no edges between these two cliques, which means that the weight-distribution bound is tight for the eigenvalue $(q^{e-1}-1)$ of $VO^-(2e,q)$.

Parabolic quadric

Let V be a $(2e)$-dimensional vector space over a finite field $\mathbb{F}_q$, where $e \geq 2$ and q is a prime power, provided with the parabolic quadratic form

$$PQ(x) = x_0^2 + x_1 x_2 + \ldots + x_{2e-1} x_{2e}.$$

The form PQ defines a bilinear form

$$\beta_{PQ}(x, y) = PQ(x + y) - PQ(x) - PQ(y).$$

A vector $x \in V$ is called isotropic if $PQ(x) = 0$. A subspace in V is called isotropic if every vector in this subspace is isotropic.

[9] B. De Bruyn, *An Introduction to Incidence Geometry*, Frontiers in Mathematics, Birkhäuser Basel (2016).

Orthogonal graphs $O(2e+1,q)$ (I)

Denote by $O(2e+1,q)$ the graph whose vertices are all isotropic (w.r.t. to the parabolic quadric) 1-dimensional subspaces on V with two vertices $[x],[y]$ being adjacent whenever one of the following three equivalent conditions holds:

- $\beta_{PQ}(x,y)=0$;
- $PQ(x+y)=0$;
- the 2-dimensional subspace including $[x]$ and $[y]$ is isotropic.

Orthogonal graphs $O(2e+1,q)$ (II)

Lemma 9

The graph $O(2e+1,q)$ is a vertex-transitive strongly regular graph with parameters

$$
\begin{aligned}
v &= \frac{q^{2e}-1}{q-1} \\
k &= \frac{q(q^{2e-2}-1)}{q-1} \\
\lambda &= \frac{q^2(q^{2e-4}-1)}{q-1} + q - 1 \\
\mu &= \frac{k}{q} = \lambda + 2
\end{aligned} \tag{4}
$$

and eigenvalues $r = q^{e-1} - 1$, $s = -q^{e-1} - 1$.

An optimal $(q^{e-1}-1)$-eigenfunction of $O(2e+1,q)$

In view of Lemmas 1 and 9, a $(q^{e-1}-1)$-eigenfunction of $O(2e+1,q)$ whose cardinality of support meets the weight-distribution bound is given by a pair of disjoint cliques of size q^{e-1}, and the cardinality of support is $2q^{e-1}$.

Consider the sets of vertices

$$V_0 = \{[(0,1,0,v_3,0,\ldots,v_{2e-1},0)] \mid v_3,\ldots,v_{2e-1},\in \mathbb{F}_q\},$$

$$V_1 = \{[(0,0,1,v_3,0,\ldots,v_{2e-1},0)] \mid v_3,\ldots,v_{2e-1},\in \mathbb{F}_q\}.$$

The sets V_0 and V_1 induce a pair of disjoint cliques of size q^{e-1}, which means that the weight-distribution bound is tight for the eigenvalue $(q^{e-1}-1)$ of $O(2e+1,q)$.

Hyperbolic quadric (revisited)

Let V be a $(2e)$-dimensional vector space over a finite field $\mathbb{F}_q$, where $e \geq 2$ and q is a prime power, provided with the hyperbolic quadratic form

$$HQ(x) = x_1x_2 + x_3x_4 + \ldots + x_{2e-1}x_{2e}.$$

The form HQ defines a bilinear form

$$\beta_{HQ}(x, y) = HQ(x + y) - HQ(x) - HQ(y).$$

A vector $x \in V$ is called isotropic if $HQ(x) = 0$. A subspace in V is called isotropic if every vector in this subspace is isotropic.

[9] B. De Bruyn, *An Introduction to Incidence Geometry*, Frontiers in Mathematics, Birkhäuser Basel (2016).

Elliptic quadric (revisited)

Let V be a $(2e)$-dimensional vector space over a finite field $\mathbb{F}_q$, where $e \geq 2$ and q is a prime power, provided with the elliptic quadratic form

$$EQ(x) = p(x_1, x_2) + x_3x_4 + \ldots + x_{2e-1}x_{2e},$$

where $p(x_1, x_2)$ is an irreducible homogeneous polynomial of degree 2 (it means that $p(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2$, $a \neq 0$, $c \neq 0$).

The form EQ defines a bilinear form

$$\beta_{EQ}(x, y) = EQ(x + y) - EQ(x) - EQ(y).$$

A vector $x \in V$ is called isotropic if $EQ(x) = 0$. A subspace in V is called isotropic if every vector in this subspace is isotropic.

[9] B. De Bruyn, *An Introduction to Incidence Geometry*, Frontiers in Mathematics, Birkhäuser Basel (2016).

Orthogonal graphs $O^{\varepsilon}(2e, q)$ (I)

Denote by $O^{\varepsilon}(2e, q)$ ($\varepsilon = 1$ or -1) the graph whose vertices are all isotropic (w.r.t. to the hyperbolic quadric if $\varepsilon = 1$ and elliptic quadric if $\varepsilon = -1$) 1-dimensional subspaces on V with two vertices $[x], [y]$ being adjacent whenever one of the following three equivalent conditions holds:

- $\beta_{HQ}(x, y) = 0$ (respectively, $\beta_{EQ}(x, y) = 0$);
- $HQ(x + y) = 0$ (respectively, $EQ(x + y) = 0$);
- the 2-dimensional subspace including $[x]$ and $[y]$ is isotropic.

Orthogonal graphs $O^{\varepsilon}(2e, q)$ (II)

Lemma 10

The graph $O^{\varepsilon}(2e, q)$ is a vertex-transitive strongly regular graph with parameters

$$\begin{aligned} v &= \frac{q^{2e}-1}{q-1} + \varepsilon q^{e-1} \\ k &= \frac{q(q^{2e-2}-1)}{q-1} + \varepsilon q^{e-1} \\ \lambda &= k - q^{2e-3} - 1 \\ \mu &= \frac{k}{q} \end{aligned} \tag{5}$$

and eigenvalues $\theta_1 = \varepsilon q^{e-1} - 1$, $\theta_2 = -\varepsilon q^{e-2} - 1$.

An optimal $(q^{e-1} - 1)$-eigenfunction of $O^+(2e, q)$

In view of Lemmas 1 and 10, a $(q^{e-1} - 1)$-eigenfunction of $O^+(2e, q)$ whose cardinality of support meets the weight-distribution bound is given by a pair of disjoint cliques of size q^{e-1}, and the cardinality of support is $2q^{e-1}$.

Consider the sets of vertices

$$V_0 = \{[(1, 0, v_3, 0, \ldots, v_{2e-1}, 0)] \mid v_3, \ldots, v_{2e-1}, \in \mathbb{F}_q\},$$

$$V_1 = \{[(0, 1, v_3, 0, \ldots, v_{2e-1}, 0)] \mid v_3, \ldots, v_{2e-1}, \in \mathbb{F}_q\}.$$

The sets V_0 and V_1 induce a pair of disjoint cliques of size q^{e-1}, which means that the weight-distribution bound is tight for the eigenvalue $(q^{e-1} - 1)$ of $O^+(2e, q)$.

An optimal $(q^{e-2}-1)$-eigenfunction of $O^-(2e,q)$

In view of Lemmas 1 and 10, a $(q^{e-2}-1)$-eigenfunction of $O^-(2e,q)$ whose cardinality of support meets the weight-distribution bound is given by a pair of disjoint cliques of size q^{e-2}, and the cardinality of support is $2q^{e-2}$.

Consider the sets of vertices

$$V_0 = \{[(0,0,1,0,v_5,0,\ldots,v_{2e-1},0)] \mid v_5,\ldots,v_{2e-1},\in \mathbb{F}_q\},$$

$$V_1 = \{[(0,0,0,1,v_5,0,\ldots,v_{2e-1},0)] \mid v_5,\ldots,v_{2e-1},\in \mathbb{F}_q\}.$$

The sets V_0 and V_1 induce a pair of disjoint cliques of size q^{e-2}, which means that the weight-distribution bound is tight for the eigenvalue $(q^{e-2}-1)$ of $O^-(2e,q)$.

Symplectic graphs $SP(2e, q)$ (I)

Let V be a $(2e)$-dimensional vector space over a finite field $\mathbb{F}_q$, where $e \geq 2$ and q is a prime power. For any nonzero $v \in V$, denote by $[v]$ the 1-dimensional subspace generated by v.
Let

$$K = \begin{pmatrix} 0 & I^{(e)} \\ -I^{(e)} & 0 \end{pmatrix}.$$

The symplectic graph $Sp(2e, q)$ relative to K over $\mathbb{F}_q$ is the graph with the set of 1-dimensional subspaces of V as its vertex set and the adjacency defined by

$[v] \sim [u]$ if and only if $vKu^t = 0$ for 1-dimensional subspaces [v], [u].

Equivalently, for arbitrary non-zero vectors $v = (v_1, \ldots, v_e, v_{e+1}, \ldots, v_{2e})$ and $u = (u_1, \ldots, u_e, u_{e+1}, \ldots, u_{2e})$, the vertices $[v]$ and $[u]$ are adjacent if and only if

$$(v_1 u_{e+1} + \ldots + v_e u_{2e}) - (v_{e+1} u_1 + \ldots + v_{2e} u_e) = 0.$$

Symplectic graphs $SP(2e, q)$ (II)

Lemma 11

The graph $SP(2e, q)$ is a vertex-transitive strongly regular graph with parameters

$$\begin{aligned} v &= \frac{q^{2e}-1}{q-1} \\ k &= \frac{q(q^{2e-2}-1)}{q-1} \\ \lambda &= \frac{q^2(q^{2e-4}-1)}{q-1} + q - 1 \\ \mu &= \frac{k}{q} = \lambda + 2 \end{aligned} \tag{6}$$

and eigenvalues $r = q^{e-1} - 1$, $s = -q^{e-1} - 1$.

An optimal $(q^{e-1}-1)$-eigenfunction of $SP(2e,q)$

In view of Lemmas 1 and 11, a $(q^{e-1}-1)$-eigenfunction of $SP(2e,q)$ whose cardinality of support meets the weight-distribution bound is given by a pair of disjoint cliques of size q^{e-1}, and the cardinality of support is $2q^{e-1}$.

$$V_0 = \{[(0, v_2, \ldots, v_e, 1, 0, \ldots, 0)] \mid v_2, \ldots, v_e \in \mathbb{F}_q\},$$

$$V_1 = \{[(1, v_2, \ldots, v_e, 1, 0, \ldots, 0)] \mid v_2, \ldots, v_e \in \mathbb{F}_q\}.$$

The sets V_0 and V_1 induce a pair of disjoint cliques of size q^{e-1}, which means that the weight-distribution bound is tight for the eigenvalue $(q^{e-1}-1)$ of $SP(2e,q)$.

Hermitian form

Let V be an n-dimensional vector space over a finite field $\mathbb{F}_q$, where q is a square. The Hermitian form on V is the mapping

$$H(x,y) = x_1 y_1^{\sqrt{q}} + \ldots + x_n y_n^{\sqrt{q}}.$$

A vector $x \in V$ is called isotropic if

$$H(x,x) = x_1^{\sqrt{q}+1} + \ldots + x_n^{\sqrt{q}+1} = 0.$$

A subspace in V is called isotropic if every vector in this subspace is isotropic.

Unitary graphs $U(n, q)$

Denote by $U(n, q)$ the graph whose vertices are all isotropic 1-dimensional subspaces on V with two vertices $[x], [y]$ being adjacent whenever one of the following two equivalent conditions holds:

- $H(x, y) = 0$;
- the 2-dimensional subspace including $[x]$ and $[y]$ is isotropic.

Unitary graphs $U(2e, q)$

Lemma 12

The graph $U(2e, q)$ is a vertex-transitive strongly regular graph with parameters

$$\begin{aligned} v &= \frac{(q^e - 1)(q^{e-\frac{1}{2}} + 1)}{q - 1} \\ k &= \frac{q(q^{e-1} - 1)(q^{e-\frac{3}{2}} + 1)}{q - 1} \\ \lambda &= \frac{q^2(q^{e-2} - 1)(q^{e-\frac{5}{2}} + 1)}{q - 1} + q - 1 \\ \mu &= \frac{k}{q} \end{aligned} \tag{7}$$

and eigenvalues $r = q^{e-1} - 1$, $s = -q^{e-\frac{3}{2}} - 1$.

Unitary graphs $U(2e+1, q)$

Lemma 13

The graph $U(2e+1, q)$ is a vertex-transitive strongly regular graph with parameters

$$\begin{aligned} v &= \frac{(q^e-1)(q^{e+\frac{1}{2}}+1)}{q-1} \\ k &= \frac{q(q^{e-1}-1)(q^{e-\frac{1}{2}}+1)}{q-1} \\ \lambda &= \frac{q^2(q^{e-2}-1)(q^{e-\frac{3}{2}}+1)}{q-1}+q-1 \\ \mu &= \frac{k}{q} \end{aligned} \tag{8}$$

and eigenvalues $r = q^{e-1}-1$, $s = -q^{e-\frac{1}{2}}-1$.

An optimal $(q^{e-1}-1)$-eigenfunction of $U(n,q)$ in the case of odd q

Let β be a primitive element in $\mathbb{F}_q$. Put $\gamma = \beta^{\frac{\sqrt{q}-1}{2}}$, which means that $\gamma^{\sqrt{q}+1} = -1$.
If $n = 2e$, consider the sets of vertices

$$V_0 = \{[(v_1, v_1\gamma, v_3, v_3\gamma, \dots, v_{2e-3}, v_{2e-3}\gamma, \gamma, 1)]\},$$

$$V_1 = \{[(v_1, v_1\gamma, v_3, v_3\gamma, \dots, v_{2e-3}, v_{2e-3}\gamma, -\gamma, 1)]\};$$

if $n = 2e+1$, consider the sets of vertices

$$V_0 = \{[(v_1, v_1\gamma, v_3, v_3\gamma, \dots, v_{2e-3}, v_{2e-3}\gamma, \gamma, 1, 0)]\},$$

$$V_1 = \{[(v_1, v_1\gamma, v_3, v_3\gamma, \dots, v_{2e-3}, v_{2e-3}\gamma, -\gamma, 1, 0)]\},$$

where in all cases $v_1, v_3, \dots, v_{2e-3}$ run over $\mathbb{F}_q$ independently.

The sets V_0 and V_1 induce a pair of disjoint cliques of size q^{e-1}, which means that the weight-distribution bound is tight for the eigenvalue $(q^{e-1}-1)$ of $U(n,q)$ in the case of odd q.

An optimal $(q^{e-1}-1)$-eigenfunction of $U(n,q)$ in the case of even q

The norm mapping $\mathbb{F}_q^* \mapsto \mathbb{F}_{\sqrt{q}}^*$ is a homomorphism defined by the rule $\delta \longrightarrow \delta^{\sqrt{q}+1}$. Note that there are exactly $\sqrt{q}+1$ elements with norm 1. Let α be an element with norm 1, $\alpha \neq 1$.

If $n = 2e$, consider the sets of vertices

$$V_0 = \{[(v_1, v_1, v_3, v_3, \dots, v_{2e-3}, v_{2e-3}, 1, 1)] \mid v_1, v_3, \dots, v_{2e-3} \in \mathbb{F}_{q^2}\},$$

$$V_1 = \{[(v_1, v_1, v_3, v_3 \dots, v_{2e-3}, v_{2e-3}, \alpha, 1)] \mid v_1, v_3, \dots, v_{2e-3} \in \mathbb{F}_{q^2}\};$$

if $n = 2e+1$, consider the sets of vertices

$$V_0 = \{[(v_1, v_1, v_3, v_3, \dots, v_{2e-3}, v_{2e-3}, 1, 1, 0)] \mid v_1, v_3, \dots, v_{2e-3} \in \mathbb{F}_{q^2}\},$$

$$V_1 = \{[(v_1, v_1, v_3, v_3, \dots, v_{2e-3}, v_{2e-3}, \alpha, 1, 0)] \mid v_1, v_3, \dots, v_{2e-3} \in \mathbb{F}_{q^2}\}.$$

The sets V_0 and V_1 induce a pair of disjoint cliques of size q^{e-1}, which means that the weight-distribution bound is tight for the eigenvalue $(q^{e-1}-1)$ of $U(n,q)$ in the case of even q.

Thank you for your attention!