# Cayley Neumaier graphs with a spread given by the cosets of a subgroup

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# Definitions

We consider undirected graphs without loops and multiple edges.

A $k$-regular graph on $v$ vertices is called edge-regular with parameters $(v, k, \lambda)$ if every pair of adjacent vertices has $\lambda$ common neighbours.

An edge-regular graph with parameters $(v, k, \lambda)$ is called strongly regular with parameters $(v, k, \lambda, \mu)$ if every pair of distinct non-adjacent vertices has $\mu$ common neighbours.

A clique in a regular graph is called $m$-regular if every vertex that doesn't belong to the clique is adjacent to precisely $m$ vertices from the clique. For an $m$-regular clique, the number $m$ is called the nexus.

# A question by Neumaier

For the clique number $\omega(\Gamma)$ of a strongly regular graph $\Gamma$, the Delsarte-Hoffman bound holds:

$$\omega(\Gamma) \leq 1 - \frac{k}{\theta_{\min}},$$

where $\theta_{\min}$ is the smallest eigenvalue of $\Gamma$.

A clique in a strongly regular graph is regular if and only if it has $1 - \frac{k}{\theta_{\min}}$ vertices; such a clique is called a Delsarte clique.

In 1981, Neumaier proved [1] that an edge-regular graph which is vertex-transitive, edge-transitive, and has a regular clique is strongly regular.

Neumaier then asked: "**Is it true that every edge-regular graph with a regular clique is strongly regular?**"

[1] A. Neumaier, *Regular Cliques in graphs and Special $1\frac{1}{2}$-designs*, Finite Geometries and Designs, London Mathematical Society Lecture Note Series, 245–259 (1981).

# Neumaier graphs

A non-complete edge-regular graph with parameters $(v, k, \lambda)$ containing an $m$-regular $s$-clique is said to be a Neumaier graph with parameters $(v, k, \lambda; m, s)$.

It follows that if a Neumaier graph with parameters $(v, k, \lambda; m, s)$ has an $m$-regular clique of size $s$, then all cliques of size $s$ in this graph are $m$-regular.

Thus, the notion of a Neumaier graph is a generalisation of the notion of a strongly regular graph with a Delsarte clique.

For a Neumaier graph with parameters $(v, k, \lambda; m, s)$, the number $m$ is called the nexus of this graph.

A Neumaier graph that is not strongly regular is said to be a strictly Neumaier graph.

For a Neumaier graph, a spread is a partition of the vertex set into regular cliques.

# Outline

- Strictly Neumaier graphs with nexus 1
  - A first construction by Greaves & Koolen;
  - Another construction by Greaves & Koolen;
  - A generalisation of Greaves & Koolen's constructions and application of the Wang-Qiu-Hu switching to it;
  - An infinite class of strictly Neumaier graphs based on the general construction
- Strictly Neumaier graphs with nexus greater than 1
  - Determination of the smallest strictly Neumaier graph and a construction of strictly Neumaier graphs with $2^i$-regular cliques, for every positive integer $i$;
- Cayley Neumaier graphs with a spread given by the cosets of a subgroup
  - Necessary and sufficient conditions;
  - Algorithm for enumeration and numerical results.

# The first construction of strictly Neumaier graphs

In [2], Greaves and Koolen constructed an infinite family of strictly Neumaier graphs with 1-regular cliques.

For positive integers $\ell$, $m$ and an odd prime power $q$, consider the group $G_{\ell,m,q} := \mathbb{Z}_\ell \oplus \mathbb{Z}_2^m \oplus \mathbb{F}_q$. Put

$$S_0 := \{(x, y, 0) \mid x \in \mathbb{Z}_\ell, y \in \mathbb{Z}_2^m, (x, y) \neq (0, 0)\}$$

Let $\pi : \mathbb{Z}_2^m \setminus \{0\} \to \{0, \ldots, 2m - 2\}$ be a bijection and $\rho$ be a primitive element of $\mathbb{F}_q$.

For each $y \in \mathbb{Z}_2^m \setminus \{0\}$, define

$$S_{y,\pi} := \{(0, y, \rho^j) \mid \pi(y) \equiv j \ (mod\ 2^m - 1)\}$$

Consider the parametrised Cayley graph $\mathrm{Cay}(G_{\ell,m,q}, S(\pi))$, where

$$S(\pi) := S_0 \cup \bigcup_{y \,\in\, \mathbb{Z}_2^m \setminus \{0\}} S_{y,\pi}$$

[2] G. R. W. Greaves, J. H. Koolen, *Edge-regular graphs with regular cliques*, Europ. J. Combin., 71, 194–201 (2018).

# The first construction of strictly Neumaier graphs

Let $q = 2nr + 1$ for some positive integer $r$. For each $i \in \{0, \dots, n-1\}$, define the cyclotomic class

$$C_q^n(i) := \{\rho^{nj+i} \mid j \in 0, \dots, 2r-1\}.$$

For $a, b \in \{0, \dots, n-1\}$, define the cyclotomic number

$$c_q^n(a, b) := |C_q^n(a) + 1 \cap C_q^n(b)|.$$

Put $c := c_q^n(a, b)$ and $\ell := (1 + c)/2$.

**Theorem ([2, Theorem 3.6, Corollary 4.4])**

*Let $q \equiv 1 \pmod 6$, $c$ be odd and $\pi : \mathbb{Z}_2^2 \setminus \{0\} \to \{0, 1, 2\}$ be a bijection. Then $\mathrm{Cay}(G_{\ell,2,q}, S(\pi))$ is a strictly Neumaier graph with parameters $(4\ell q, 4\ell - 2 + q, 4\ell - 2; 1, 4\ell)$.*

[2] G. R. W. Greaves, J. H. Koolen, *Edge-regular graphs with regular cliques*, Europ. J. Combin., 71, 194–201 (2018).

# Notes on the first construction

- Set $q := 7^a$, where $a \not\equiv 0 \ (mod\ 3)$. Then $\mathrm{Cay}(G_{\ell,2,q}, S(\pi))$ is a strictly Neumaier graph with parameters

$$(4\ell q, 4\ell - 2 + q, 4\ell - 2; 1, 4\ell).$$

  In particular, if $a = 1$, then we have a strictly Neumaier graph with parameters $(28, 9, 2; 1, 4)$. This graph is the smallest example from [2].
- $\mathrm{Cay}(G_{\ell,2,q}, S(\pi))$ has a spread of size $q$ given by the cosets of the subgroup $\{(x, y, 0) \mid x \in \mathbb{Z}_\ell, y \in \mathbb{Z}_2^m\}$.

[2] G. R. W. Greaves, J. H. Koolen, *Edge-regular graphs with regular cliques*, Europ. J. Combin., 71, 194–201 (2018).

# Four strictly Neumaier graphs on 24 vertices

Gavrilyuk and Goryainov then searched for examples in a collection of known Cayley-Deza graphs [3] and found four more strictly Neumaier graphs with parameters $(24, 8, 2; 1, 4)$.

In [4], Greaves and Koolen found 'another' infinite family of strictly Neumaier graphs, which contains one of the four graphs on 24 vertices.

[3] S. V. Goryainov, L. V. Shalaginov, *Cayley-Deza graphs with fewer than 60 vertices*, Siberian Electronic Mathematical Reports, 11, 268–310 (2014) (in Russian).

[4] G. R. W. Greaves, J. H. Koolen, *Another construction of edge-regular graphs with regular cliques*, Discrete Mathematics, 342, Issue 10, (2019) 2818–2820.

# Antipodal distance-regular graphs

A graph $\Gamma$ of diameter $d$ is called distance-regular if, for any two vertices $x, y \in V(\Gamma)$, the number of vertices at distance $i$ from $x$ and distance $j$ from $y$ depends only on $i, j$, and the distance from $x$ to $y$. It is clear that distance regular graphs are edge-regular.

A distance-regular graph $\Gamma$ of diameter $d$ is called $a$-antipodal if the relation of being at distance $d$ or distance 0 is an equivalence relation on the vertices of $\Gamma$ with equivalence classes of size $a$.

# The second construction of strictly Neumaier graphs

Let $\Gamma$ be an $a$-antipodal distance-regular graph of diameter 3 with edge-regular parameters $(v, k, \lambda)$ such that $a$ is a proper divisor of $\lambda + 2$.

Put $t = \frac{\lambda+2}{a}$ and take $t$ disjoint copies $\Gamma^{(1)}, \ldots, \Gamma^{(t)}$ of $\Gamma$.

For every antipodal class $H$ in $\Gamma$, take the corresponding antipodal classes $H^{(1)}, \ldots, H^{(t)}$ in $\Gamma^{(1)}, \ldots, \Gamma^{(t)}$, respectively, and connect any two vertices from $H^{(1)} \cup \ldots \cup H^{(t)}$ to form a 1-regular clique of size $at$.

Denote by $F_t(\Gamma)$ the resulting graph.

## Theorem ([4])

*The graph $F_t(\Gamma)$ is a strictly Neumaier graph having parameters $(tv, k + at - 1, \lambda; 1, at)$ and containing a spread.*

[4] G. R. W. Greaves, J. H. Koolen, *Another construction of edge-regular graphs with regular cliques*, Discrete Mathematics, 342, Issue 10, (2019) 2818–2820.

# Notes on the second construction

- In particular, if $\Gamma$ is the icosahedron, then $a = 2$, $\lambda = 2$, $t = 2$ and $F_2(\Gamma)$ is one of the four strictly Neumaier graphs with parameters $(24, 8, 2; 1, 4)$ found in [3].
- The other three graphs can be obtained in a similar way by choosing an appropriate matching of the antipodal classes in the two copies of the icosahedrons.

[3] S. V. Goryainov, L. V. Shalaginov, *Cayley-Deza graphs with fewer than 60 vertices*, Siberian Electronic Mathematical Reports, 11, 268–310 (2014) (in Russian).

# Perfect codes in graphs

Let $\Gamma$ be a simple undirected graph and $e \geq 1$ be an integer.

The ball with radius $e$ and centre $u \in V(\Gamma)$ is the set of vertices of $\Gamma$ with distance at most $e$ to $u$ in $\Gamma$.

A subset $C$ of $V(\Gamma)$ is called a perfect $e$-code in $\Gamma$ if the balls with radius $e$ and centres in $C$ form a partition of $V(\Gamma)$.

In particular, a perfect 1-code is a subset of vertices $C$ such that every vertex not in $C$ is adjacent to a unique element of $C$.

# A generalisation of the two constructions

Let $\Gamma^{(1)}, \dots, \Gamma^{(t)}$ be edge-regular graphs with parameters $(v, k, \lambda)$, and such that each $\Gamma^{(i)}$ has a partition of its vertices into perfect 1-codes of size $a$, where $a$ is a proper divisor of $\lambda + 2$. Further, we define $t = (\lambda + 2)/a$.

For any $\ell \in \{1, \dots, t\}$, let $H_1^{(\ell)}, \dots, H_{v/a}^{(\ell)}$ denote the perfect 1-codes that partition the vertex set of $\Gamma^{(\ell)}$.

If $t \geq 2$, we also take a $(t-1)$-tuple of permutations from $\mathrm{Sym}(\{1, \dots, v/a\})$, denoted by $\Pi = (\pi_2, \dots, \pi_t)$.

Using these graphs and the tuple $\Pi$, we define the graph $F_\Pi(\Gamma^{(1)}, \dots, \Gamma^{(t)})$ as follows.

1. Take the disjoint union of the graphs $\Gamma^{(1)}, \dots, \Gamma^{(t)}$.
2. For any $i \in \{1, \dots, v/a\}$, add an edge between any two distinct vertices from $H_i^{(1)} \cup H_{\pi_2(i)}^{(2)} \cup \dots \cup H_{\pi_t(i)}^{(t)}$ (which forms a 1-regular clique of size $at$).

# A generalisation of the two constructions

## Theorem ([5])

*The following statements hold.*

1. $F_{\Pi}(\Gamma^{(1)}, \dots, \Gamma^{(t)})$ *has (a spread of)* 1*-regular cliques, each of size* $\lambda + 2$*;*
2. $F_{\Pi}(\Gamma^{(1)}, \dots, \Gamma^{(t)})$ *is an edge-regular graph with parameters* $(vt, k + \lambda + 1, \lambda)$.

[5] R. J. Evans, S. Goryainov, E. V. Konstantinova, A. D. Mednykh, *A general construction of strictly Neumaier graphs and a related switching*, September 2021. `https://arxiv.org/abs/2109.13884`

# A generalisation of the two constructions

## Remark

*We note that a converse to the theorem above is also true. Let* $\Gamma \in NG(v, k, \lambda; 1, s)$ *have a spread of* 1*-regular cliques. The graph* $\Gamma^\circ$ *created by removing edges from the cliques of this spread in* $\Gamma$ *is also edge-regular by Soicher [6, Theorem 6.1]. It also follows that connected components of* $\Gamma^\circ$ *can be partitioned into perfect* 1*-codes, and the parameters have the same restrictions as the conditions of the statement of the theorem above.*

[6] L. H. Soicher, *On cliques in edge-regular graphs*, *Journal of Algebra*, 421, 260–267 (2015). https://doi.org/10.1016/j.jalgebra.2014.08.028

# A corollary

In the case for which $t = 1$ can occur, the construction can result in a strictly Neumaier graph. However, this is not necessarily true in all cases when $t = 1$. The following Corollary shows that for $t \geq 2$, our construction always results in a strictly Neumaier graph.

## Corollary ([5])

*Let* $\Gamma^{(1)}, \ldots, \Gamma^{(t)}$ *be non-complete edge-regular graphs with parameters* $(v, k, \lambda)$ *and let* $\Pi = (\pi_2, \ldots, \pi_t)$ *be a* $(t-1)$*-tuple of permutations from* $Sym(\{1, \ldots, v/a\})$.
*Further, suppose that each graph* $\Gamma^{(\ell)}$ *has a partition of its vertices into perfect* 1*-codes* $H_1^{(\ell)}, \ldots, H_{v/a}^{(\ell)}$, *each of size* $a$, *where* $a$ *is a proper divisor of* $\lambda + 2$ *and* $t = (\lambda + 2)/a$.
*If* $t \geq 2$, *then* $F_\Pi(\Gamma^{(1)}, \ldots, \Gamma^{(t)})$ *is a strictly Neumaier graph.*

[5] R. J. Evans, S. Goryainov, E. V. Konstantinova, A. D. Mednykh, *A general construction of strictly Neumaier graphs and a related switching*, September 2021. https://arxiv.org/abs/2109.13884

# Notes on the generalisation

- Non-isomorphic Taylor graphs with the same parameters give many new examples in the case $t \geq 2$.
- The four strictly Neumaier graphs on 24 vertices from [3] are given by a pair of icosahedrons, and the only difference between them is the choice of the permutation that matches the antipodal classes.
- The generalised construction covers both constructions from [2] and [4] (the cases $t = 1$ and $t \geq 2$, respectively).
- For $t = 1$ we can construct three new strictly Neumaier graphs: with parameters $(28, 9, 2; 1, 4)$, $(40, 12, 2; 1, 4)$ and $(65, 16, 3; 1, 5)$; eight graphs with parameters $(78, 17, 4; 1, 6)$.

[2] G. R. W. Greaves, J. H. Koolen, *Edge-regular graphs with regular cliques*, Europ. J. Combin., 71, 194–201 (2018).

[3] S. V. Goryainov, L. V. Shalaginov, *Cayley-Deza graphs with fewer than 60 vertices*, Sibirskie Èlektronnye Matematicheskie Izvestiya, 11, 268–310 (2014).

[4] G. R. W. Greaves, J. H. Koolen, *Another construction of edge-regular graphs with regular cliques*, Dis. Math., 342, Issue 10, (2019) 2818–2820.

# Notes on the generalisation

- In [7], Corollary was independently proved and an infinite class of strictly Neumaier graphs based on the general construction was obtained.

[7] A. Abiad, W. Castryck, M. De Boeck, J. H. Koolen, S. Zeijlemaker, *An infinite class of Neumaier graphs and non-existence results*, Journal of Combinatorial Theory, Series A Volume 193, January 2023, 105684.
https://doi.org/10.1016/j.jcta.2022.105684

# Examples from infinite edge-regular lattices (see [5])

Eisenstein integers are the complex numbers of the form $\mathbb{Z}[\omega] = \{b + c\omega : b, c \in \mathbb{Z}\}$, where $\omega = \frac{-1+i\sqrt{3}}{2}$. They form a ring with respect to usual addition and multiplication.

The norm mapping $N : \mathbb{Z}[\omega] \mapsto \mathbb{N} \cup \{0\}$ is defined as follows. For an Eisenstein integer $b + c\omega$, $N(b + c\omega) = b^2 + c^2 - bc$ holds. The norm mapping $N$ is known to be multiplicative.

It is well-known that $\mathbb{Z}[\omega]$ forms an Euclidean domain (in particular, a principal ideal domain).

The units of $\mathbb{Z}[\omega]$ are $\{\pm 1, \pm\omega, \pm\omega^2\}$. The natural geometrical interpretation of Eisenstein integers is the 6-regular triangular grid in the complex plane.

If it does not lead to a contradiction, we use the same notation $\mathbb{Z}[\omega]$ for the triangular grid.

[5] R. J. Evans, S. Goryainov, E. V. Konstantinova, A. D. Mednykh, *A general construction of strictly Neumaier graphs and a related switching*, September 2021. https://arxiv.org/abs/2109.13884

# Examples from infinite edge-regular lattices

The grid $\mathbb{Z}[\omega]$ has exactly six elements of norm 7; these are $\{\pm(1+3\omega), \pm(3+2\omega), \pm(2-\omega)\}$. Consider the ideal $I$ generated by an element of norm 7 (say, by the element $2-\omega$). The elements of $I$ form a perfect 1-code in the triangular grid. Note that $I$ is an additive subgroup of index 7 in $\mathbb{Z}[\omega]$; we denote it by $I^+$. The seven cosets $\mathbb{Z}[\omega]/I^+$ give a partition of the triangular grid into seven perfect 1-codes. Take the following two additive subgroups of $\mathbb{Z}[\omega]$:

$$T_1 := \{2(-2+\omega)x + 14y \mid x, y \in \mathbb{Z}\},$$

$$T_2 := \{(5+\omega)x + 28y \mid x, y \in \mathbb{Z}\}.$$

Since $-2+\omega$, 7 and $5+\omega$ are divisible by $2-\omega$, a generator of $I$, we have that $T_1$ and $T_2$ are subgroups in $I^+$. Note that there exists a block of four balls of radius 1 such that the additive shifts of the block by the elements of $T_1$ and $T_2$ give two tessellations of $\mathbb{Z}[\omega]$.

# Examples from infinite edge-regular lattices

Consider the quotient groups

$$G_1 := \mathbb{Z}[\omega]/T_1 \text{ and } G_2 := \mathbb{Z}[\omega]/T_2,$$

where $G_1 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{14}$ and $G_2 \cong \mathbb{Z}_{28}$. Define two Cayley graphs

$$\Delta_1 := \mathrm{Cay}(G_1, \{\pm(1+T_1), \pm(\omega+T_1), \pm(\omega^2+T_1)\}),$$

$$\Delta_2 := \mathrm{Cay}(G_2, \{\pm(1+T_2), \pm(\omega+T_2), \pm(\omega^2+T_2)\}).$$

Note that $\Delta_1$ and $\Delta_2$ can be interpreted as quotient graphs of the triangular grid by $T_1$ and $T_2$, respectively. Each of the graphs $\Delta_1$ and $\Delta_2$ is edge-regular with parameters $(28, 6, 2)$ and admits a partition into perfect 1-codes of size $a = 4$; these partitions are given by the original partition of the triangular grid into perfect 1-codes. We then apply the general construction, which gives two strictly Neumaier graphs with parameters $(28, 9, 2; 1, 4)$. The graph obtained from $\Delta_1$ is isomorphic to the smallest graph from the first Greaves & Koolen's construction, and the graph obtained from $\Delta_2$ is new.

# More infinite edge-regular lattices

In the following, we consider countably infinite graphs with vertices consisting of elements of the vector space $\mathbb{R}^n$, for some integer $n \geq 3$. The elements of $\mathbb{R}^n$ are called ($n$-dimensional) vectors, and we identify the elements with their coordinates with respect to the standard basis of $\mathbb{R}^n$.

Let $x \in \mathbb{R}^n$. For a set $A \subseteq \mathbb{R}$, the vector $x$ is an $A$-vector if the value of all of its entries lie in $A$. The weight of $x$ is the number of its non-zero entries.

Let $n \geq 3$ be a positive integer and let $m$ be an even positive integer. Let $S_{n,m}^{(1)}$ denote the set of all $n$-dimensional $\{1,-1,0\}$-vectors of weight $m$ whose sum of coordinates is zero. Let $S_{n,m}^{(2)}$ denote the set of all $n$-dimensional $\{1,-1,0\}$-vectors of weight $m$. Let $G_{n,m}^{(1)}$ and $G_{n,m}^{(2)}$ be the groups generated by $S_{n,m}^{(1)}$ and $S_{n,m}^{(2)}$ respectively.

# More infinite edge-regular lattices

## Proposition

*For any positive even integer $m$ and any integer $n$ such that $n \geq m+1$, the following statements hold.*

1. *$G^{(1)}_{n,m}$ is equal to $G^{(1)}_{n,2}$, which consists of all $n$-dimensional vectors with integer coordinates such that the sum of coordinates is equal to $0$.*
2. *$G^{(2)}_{n,m}$ is equal to $G^{(2)}_{n,2}$, which consists of all $n$-dimensional vectors with integer coordinates such that the sum of coordinates is even.*

# More infinite edge-regular lattices

From now on, we let

$$G_n^{(1)} := G_{n,2}^{(1)},$$

$$G_n^{(2)} := G_{n,2}^{(2)},$$

and define graphs

$$\Gamma_{n,m}^{(1)} := \mathrm{Cay}(G_n^{(1)}, S_{n,m}^{(1)})$$

and

$$\Gamma_{n,m}^{(2)} := \mathrm{Cay}(G_n^{(2)}, S_{n,m}^{(2)}).$$

A graph $\Gamma$ with infinitely many vertices is edge-regular with parameters $(k, \lambda)$ if it is $k$-regular and each pair of adjacent vertices have exactly $\lambda$ common neighbours.

In the following, we show that $\Gamma_{n,m}^{(1)}$ and $\Gamma_{n,m}^{(2)}$ are infinite edge-regular graphs, and give the parameters of these graphs in terms of binomial coefficients.

# More infinite edge-regular lattices

## Proposition

*For any positive even integer $m$ and any integer $n$ such that $n \geq m+1$, the following statements hold.*

1. *The graph $\Gamma^{(1)}_{n,m}$ is an induced subgraph in $\Gamma^{(2)}_{n,m}$.*
2. *$\Gamma^{(1)}_{n,m}$ is an infinite edge-regular graph with parameters $(k_1, \lambda_1)$, such that*

$$k_1 = \binom{n}{m}\binom{m}{\frac{m}{2}},\ \lambda_1 = \sum_{i=0}^{\frac{m}{2}} \binom{\frac{m}{2}}{i}\binom{\frac{m}{2}}{\frac{m}{2}-i}\binom{n-m}{\frac{m}{2}-i}\binom{n-\frac{3m}{2}+i}{i}.$$

3. *$\Gamma^{(2)}_{n,m}$ is an infinite edge-regular graph with parameters $(k_1, \lambda_1)$, such that*

$$k_2 = 2^m\binom{n}{m},\ \ \lambda_2 = \binom{m}{\frac{m}{2}}\binom{n-m}{\frac{m}{2}}2^{\frac{m}{2}}.$$

# Comments

**Remark**
*Note that if* $n < \frac{3m}{2}$, *then* $\lambda_1 = 0$ *and* $\lambda_2 = 0$. *Otherwise,* $\lambda_1 > 0$ *and* $\lambda_2 > 0$.

**Remark**
*The generating sets* $S^{(1)}_{n,2}$ *and* $S^{(2)}_{n,2}$ *are known as root systems* $A_{n-1}$ *and* $D_n$.

$A_2$ root lattice is isomorphic to the 6-regular triangular grid.

The root lattices $A_3$ and $D_3$ are both isomorphic to the tetrahedral-octahedral honeycomb.

# More examples from infinite edge-regular lattices

In the following tables, we present the number of cases for which we find strictly Neumaier graphs using the graphs $\Gamma^{(1)}_{n,m}$ and $\Gamma^{(2)}_{n,m}$, respectively.

The first column of the tables give the corresponding value of $n$.

The second column gives the Neumaier graph parameters of the graphs we find through the construction. The last column gives the number of pairwise non-isomorphic strictly Neumaier graphs we find from the construction.

# More examples from infinite edge-regular lattices

| $n$ | parameters of SNG | # |
|---|---|---|
| 3 | $(28, 9, 2; 1, 4)$ | 2 |
| 4 | $(78, 17, 4; 1, 6)$ | $\geq 8$ |
| 5 | $(168, 27, 6; 1, 8)$ | $\geq 12$ |
| 6 | $(310, 39, 8; 1, 10)$ | $\geq 1$ |

Table: Number of strictly Neumaier graphs from quotients of $\Gamma^{(1)}_{n,2}$.

| $n$ | parameters of SNG | # |
|---|---|---|
| 3 | $(78, 17, 4; 1, 6)$ | $\geq 8$ |
| 4 | $(250, 33, 8; 1, 10)$ | $\geq 16$ |

Table: Number of strictly Neumaier graphs from quotients of $\Gamma^{(2)}_{n,2}$.

# A problem

However, we have not been able to find more examples of perfect codes and quotients of $\Gamma_{n,2}^{(1)}$ and $\Gamma_{n,2}^{(2)}$ that lead to strictly Neumaier graphs. Therefore, we ask the following.

**Problem**

*What strictly Neumaier graphs can be obtained from quotients of infinite edge-regular graphs $\Gamma_{n,m}^{(1)}$ and $\Gamma_{n,m}^{(2)}$?*

# Examples from infinite edge-regular lattices

We can also use two infinite edge-regular graphs to get a new infinite edge-regular graph by taking the Cartesian product of the graphs.

## Proposition

*Let* $\Gamma_1$ *and* $\Gamma_2$ *be two infinite edge-regular graphs with parameters* $(k_1, \lambda)$ *and* $(k_2, \lambda)$*, respectively. Then the Cartesian product of* $\Gamma_1$ *and* $\Gamma_2$ *is an edge-regular graph with parameters* $(k_1 + k_2, \lambda)$*.*

# An example from Cartesian products of infinite edge-regular lattices

Consider the Cartesian product of two 6-regular triangular grids; the resulting infinite graph is edge-regular with parameters (12,2). This graph has a subgroup perfect 1-code, and there exists an edge-regular quotient graph with parameters $(52, 12, 2)$. We then apply the general construction to this graph, which gives a strictly Neumaier graph having parameters $(52, 15, 2; 1, 4)$ and isomorphic to the second largest graph from the first Greaves & Koolen's construction. As we have seen this example using Cartesian products, we ask the following.

**Problem**

*What strictly Neumaier graphs can be obtained from quotients of Cartesian products of infinite edge-regular graphs?*

# Spectrum of a graph

The spectrum of a graph $\Gamma$ is the multiset of eigenvalues of the adjacency matrix of $\Gamma$.

Two graphs are cospectral if they have the same spectra.

The following switching (Wang-Qu-Hu switching), which produces cospectral graphs, was discovered in [8], applied in [9] to obtain new strongly regular graphs, and discussed in [10].

[8] W. Wang, L. Qiu, Y. Hu, Cospectral graphs, GM-switching and regular rational orthogonal matrices of level $p$, Linear Algebra and its Applications, Volume 563, 15 (2019), 154–177.
`https://doi.org/10.1016/j.laa.2018.10.027`

[9] F. Ihringer, A. Munemasa, New strongly regular graphs from finite geometries via switching, Linear Algebra and its Applications Volume 580, (2019), 464–474. `https://doi.org/10.1016/j.laa.2019.07.014`

[10] L. Qiu, Y. Ji, W. Wang, On a theorem of Godsil and McKay concerning the construction of cospectral graphs, Linear Algebra and its Applications, Volume 603 (2020), 265–274.
`https://doi.org/10.1016/j.laa.2020.05.025`

# Wang-Qu-Hu switching

## Lemma (WQH-switching)

*Let* $\Gamma$ *be a graph whose vertex set is partitioned as* $C_1 \cup C_2 \cup D$. *Assume that* $|C_1| = |C_2|$ *and that the induced subgraphs on* $C_1, C_2$, *and* $C_1 \cup C_2$ *are regular, where the degrees in the induced subgraphs on* $C_1$ *and* $C_2$ *are the same. Suppose that all* $x \in D$ *satisfy one of the following:*

1. $|\Gamma(x) \cap C_1| = |\Gamma(x) \cap C_2|$, *or*
2. $\Gamma(x) \cap (C_1 \cup C_2)| \in \{C_1, C_2\}$.

*Construct a graph* $\Gamma'$ *from* $\Gamma$ *by modifying the edges between* $C_1 \cup C_2$ *and* $D$ *as follows:*

$$\Gamma'(x) \cap (C_1 \cup C_2) = \begin{cases} C_1, & \text{if } \Gamma(x) \cap (C_1 \cup C_2) = C_2; \\ C_2, & \text{if } \Gamma(x) \cap (C_1 \cup C_2) = C_1; \\ \Gamma(x) \cap (C_1 \cup C_2), & \text{otherwise,} \end{cases}$$

*for all* $x \in D$. *Then* $\Gamma'$ *is cospectral with* $\Gamma$.

# WQH-switching for the general construction

## Proposition ([5])

*Let* $t \geq 2$ *and* $F_\Pi(\Gamma^{(1)}, \ldots, \Gamma^{(t)})$ *be a strictly Neumaier graph obtained from the general construction. Then for any non-empty subset* $I \subseteq \{1, \ldots, t\}$ *containing 1 and distinct* $i, j \in \{1, \ldots, v/a\}$*, the partition*

$$C_1 := \bigcup_{\ell \in I} H_i^{(\ell)}, \quad C_2 := \bigcup_{\ell \in I} H_j^{(\ell)},$$

$$D := V(F_\Pi(\Gamma^{(1)}, \ldots, \Gamma^{(t)})) \setminus (C_1 \cup C_2)$$

*satisfies the conditions of WQH-switching.*
*Moreover, we have the equality*
$(F_\Pi(\Gamma^{(1)}, \ldots, \Gamma^{(t)}))' = F_{\Pi'}(\Gamma^{(1)}, \ldots, \Gamma^{(t)})$*, where*

$$(\Pi')_r = \begin{cases} \pi_r & \text{if } r \in I \\ (i\ j) \circ \pi_r & \text{if } r \notin I \end{cases}$$

# Corollaries

## Corollary ([5])

*For any* $\Pi, \Pi'$, $(t-1)$*-tuples of elements of* $Sym(\{1,\ldots,v/a\})$, *the graphs* $F_{\Pi}(\Gamma^{(1)},\ldots,\Gamma^{(t)})$ *and* $F_{\Pi'}(\Gamma^{(1)},\ldots,\Gamma^{(t)})$ *are cospectral.*

It would be interesting to investigate how many pairwise non-isomorphic graphs can be constructed using our construction. In doing so, we may find a prolific construction of cospectral strictly Neumaier graphs. Although this has not been investigated in detail, we have already observed several pairwise non-isomorphic graphs with relatively small order.

## Corollary ([5])

*The four strictly Neumaier graphs with parameters* $(24, 8, 2; 1, 4)$ *obtained from two copies of icosahedron are cospectral.*

[5] R. J. Evans, S. Goryainov, E. V. Konstantinova, A. D. Mednykh, *A general construction of strictly Neumaier graphs and a related switching*, September 2021. https://arxiv.org/abs/2109.13884

# The smallest strictly Neumaier graph

In [11], Evans, G. and Panasenko found the smallest strictly Neumaier graph, which is a Cayley graph, has parameters $(16, 9, 4; 2, 4)$ and contains a spread given by the cosets of a subgroup.

It can be constructed by switching edges in the affine polar graph $VO^+(4, 2)$, which is isomorphic to the complement to $(4 \times 4)$-lattice.

[11] R. J. Evans, S. V. Goryainov, D. I. Panasenko, The smallest strictly Neumaier graph and its generalisations, The Electronic Journal of Combinatorics, 26(2) (2019), #P2.29.

# Strictly Neumaier graphs with $2^i$-regular cliques

We then generalised the smallest strictly Neumaier graph and, for every positive integer $i$, by switching in certain affine polar graphs, found a strictly Neumaier graph with $4^{i+1}$ vertices containing a $2^i$-regular clique and having parameters of these affine polar graphs as edge-regular graphs.

However, the graphs for $i \geq 2$ were not vertex-transitive, and it was an open question whether there exists a vertex-transitive strictly Neumaier graph with nexus greater than 1 except the smallest strictly Neumaier graph. In this project we solve this question in positive.

[11] R. J. Evans, S. V. Goryainov, D. I. Panasenko, The smallest strictly Neumaier graph and its generalisations, The Electronic Journal of Combinatorics, 26(2) (2019), #P2.29.

# Problem on strictly Neumaier graphs

In general, we are interested in the following problem.

### Problem

*For which positive integers $m$ does there exist a strictly Neumaier graph with an $m$-regular clique?*

### Remark

*All previously known strictly Neumaier graphs had regular cliques with nexus equal to a power of 2. The only known strictly Neumaier graphs having regular cliques with nexus greater than 1 were found in [11].*

Motivated by the fact that many known examples of strictly Neumaier graphs are Cayley graphs with a spread given by the cosets of a subgroup, we decided to have a general look at Cayley Neumaier graphs with a spread given by the cosets of a subgroup.

[11] R. J. Evans, S. V. Goryainov, D. I. Panasenko, The smallest strictly Neumaier graph and its generalisations, The Electronic Journal of Combinatorics, 26(2) (2019), #P2.29.

# Equitable partitions

Let $\Gamma$ be a $k$-regular graph with the vertex set $V(\Gamma)$.

Let $\Pi := (V_1, \ldots, V_t)$ be a partition of $V(\Gamma)$ into $t$ parts ($t$-partition).

The partition $\Pi$ is said to be an equitable $t$-partition if for any $i, j \in \{1, \ldots, t\}$ there is a constant $p_{ij}$ such that any vertex from the part $V_i$ is adjacent to precisely $p_{ij}$ vertices from the part $V_j$.

The square matrix $P_\Pi := (p_{ij})_{i,j=1}^{t}$ is called the quotient matrix of the equitable $t$-partition $\Pi$.

# Subgroup equitable 2-partitions in Cayley graphs

## Lemma (Evans, G., Zhao, 2022+)

*Let $G$ be a finite group, $H < G$ with cosets $Hg_1, \ldots, Hg_n$, with $g_1 = 1$. Further, let $S \subseteq G, 1 \notin S$.*
*Then $\{H, G \setminus H\}$ is an equitable 2-partition in $\Gamma = \mathrm{Cay}(G, S)$ if and only if*

1. *$S \cap H$ is closed under inversion.*
2. *There exists subsets $T_2, T_3, \ldots, T_n \subseteq S$, such that*
   - 2.1 $|T_i| = m$
   - 2.2 *$T_i$ consists of representatives of the coset $Hg_i$*
   - 2.3 *$T = \bigcup T_i$ is closed under inversion.*

*In this case, the induced subgraph $\Gamma[H]$ is $|S \cap H|$-regular with nexus $m$, and the induced subgraph $\Gamma[G \setminus H]$ is $(|S| - m)$-regular with nexus $|S \setminus H|$. In other words, we have quotient matrix*

$$A(\Gamma/H) = \begin{pmatrix} |S \cap H| & |S \setminus H| \\ m & |S| - m \end{pmatrix}$$

# Coset equitable partitions in Cayley graphs

**Corollary (Evans, G., Zhao, 2022+)**

*Suppose* $\Gamma = \mathrm{Cay}(G, S)$ *is a Cayley graph with group equitable 2-partition corresponding to* $H < G, [G : H] = n$*. Then* $\Gamma$ *has an equitable* $n$*-partition* $X$*, with the parts the cosets of the group* $H$*, and quotient matrix*

$$A(\Gamma/X) = \begin{pmatrix} |S \cap H| & m & \cdots & m \\ m & |S \cap H| & \cdots & \vdots \\ \vdots & \cdots & |S \cap H| & m \\ m & \cdots & m & |S \cap H| \end{pmatrix}$$

# Edge-regularity of special Cayley graphs with coset equitable partition

## Lemma (Evans, G., Zhao, 2022+)

*Let $G$ be a group, $H < G$ and $S \subseteq G, 1 \notin S$, where $\{H, G \setminus H\}$ is an equitable 2-partition with corresponding sets $T_2, \ldots, T_n$ and $T = \bigcup T_i$. Furthermore, assume $H^* \cap S = \emptyset$.*
*Then $\Gamma$ is edge-regular with parameters $(v, k, \lambda)$ if and only if for all $g \in T$, the condition*

$$|Tg \cap T| = \lambda$$

*holds.*

# Cayley Neumaier graphs with a spread given by the cosets of a subgroup

**Lemma (Evans, G., Zhao, 2022+)**

*Let $G$ be a group, $H < G$ and $S \subseteq G, 1 \notin S$ where $\{H, G \setminus H\}$ is an equitable 2-partition with corresponding sets $T_2, \ldots, T_n$ and $T = \bigcup T_i$. Furthermore, assume $H^* \subseteq S$, so $S = H^* \cup T$. Then $\Gamma = \mathrm{Cay}(G, S)$ is an edge-regular (and thus Neumaier) graph with parameters $(v, k, \lambda)$ if and only if*

1. *For all $h \in H^*$, $|Th \cap T| = \lambda - |H| + 2$*
2. *For all $g \in T$, $|Tg \cap T| = \lambda - 2m + 2$*

# Cayley Neumaier graphs with a spread given by the cosets of a subgroup

**Corollary (Evans, G., Zhao, 2022+)**

*Such a Neumaier graph* $\Gamma = \mathrm{Cay}(G, S)$ *has parameters*

$$v = ns$$
$$k = s - 1 + (n-1)m$$
$$\lambda = s - 2 + \frac{(n-1)m(m-1)}{(s-1)}$$

*where* $s = |H|$.

# The smallest strictly Neumaier graph as a Cayley graph

- $G = C_2 \times C_8$
  $H = \langle (1,2) \rangle$,
  $T_2 = \{(0,1),(1,7)\}$,
  $T_3 = \{(0,2),(0,6)\}$,
  $T_4 = \{(0,7),(1,1)\}$.
- $G = D_{16} = \langle a, b | a^8 = b^2 = 1, bab = a^{-1} \rangle$,
  $H = \langle a^2 \rangle, \{1, a^4, a^2b, a^6b\}$,
  $S = \{a, a^2, a^4, a^6, a^7, ab, a^2b, a^6b, a^7b\}$.
- $G = C_2 \times D_8 = \langle 1 \rangle \times \langle a, b | a^4 = b^2 = baba = 1 \rangle$,
  $H = \langle (0,a) \rangle, \langle (1,a) \rangle$,
  $S = \{(0,a),(0,a^2),(0,a^3),(0,a^2b),(0,a^3b),(1,a),(1,a^3),$
  $(1,a^2b),(1,a^3b)\}$

In all cases, $\Gamma = \mathrm{Cay}(G, H^* \cup T)$ is the smallest strictly Neumaier graph, with a spread given by the cosets of $H$ and parameters $(16, 9, 4; 2, 4)$.

# The four graphs from the icosahedron as Cayley graphs

1. $G = S_4, H = \langle(1,3,2,4)\rangle, T_2 = \{(1,2,3)\}, T_3 = \{(1,3,2)\}, T_4 = \{(1,4,2)\}, T_5 = \{(1,3)(2,4)\}, T_6 = \{(1,2,4)\}$
2. $G = S_4, H = \langle(1,2),(3,4)\rangle, T_2 = \{(1,3,4)\}, T_3 = \{(1,4)(2,3),\}, T_4 = \{(2,4,3),\}, T_5 = \{(1,4,3),\}, T_6 = \{(2,3,4)\}$
3. $G = C_2 \times A_4, H = \langle(1,(1,2)(3,4)),(1,(1,3)(2,4))\rangle, T_2 = \{(0,(1,2,4)),\}, T_3 = \{(0,(1,3,4)),\}, T_4 = \{(0,(1,3)(2,4))\}, T_5 = \{(0,(1,4,2))\}, T_6 = \{(0,(1,4,3))\}$
4. $G = C_2 \times A_4, H = \langle(0,(1,4)(2,3)),(1,())\rangle, T_2 = \{(0,(1,2,3))\}, T_3 = \{(0,(2,3,4))\}, T_4 = \{(0,(2,4,3))\}, T_5 = \{(0,(1,3,2))\}, T_6 = \{(0,(1,2)(3,4))\}$

Then for each case, $\Gamma = \mathrm{Cay}(G, H^* \cup T)$ is a strictly Neumaier graph with with a spread given by the cosets of $H$ and parameters $(24, 8, 2; 1, 4)$.

# Two strictly Neumaier graphs with parameters $(28, 9, 2; 1, 4)$

1. $G = C_{28} = C_4 \times C_7, H = \langle(1,0)\rangle, T_2 = \{(1,1)\}, T_3 = \{(3,6)\}, T_4 = \{(1,3)\}, T_5 = \{(3,4)\}, T_6 = \{(0,2)\}, T_7 = \{(0,5)\}$
2. $G = C_{28} = C_4 \times C_7, H = \langle(1,0)\rangle, T_2 = \{(3,1)\}, T_3 = \{(1,6)\}, T_4 = \{(3,3)\}, T_5 = \{(1,4)\}, T_6 = \{(0,2)\}, T_7 = \{(0,5)\}$
3. $G = C_2 \times C_{14} = C_2 \times C_2 \times C_7, H = \langle(1,0,0),(0,1,0)\rangle, T_2 = \{(1,0,1)\}, T_3 = \{(1,0,4)\}, T_4 = \{(0,1,2)\}, T_5 = \{(0,1,5)\}, T_6 = \{(1,1,3)\}, T_7 = \{(1,1,6)\}$

Then for the cases 1,2 we find that the graphs $\text{Cay}(G, H^* \cup T)$ are isomorphic with different generating sets.

In all cases above, $\Gamma = \text{Cay}(G, H^* \cup T)$ is a strictly Neumaier graph with with a spread given by the cosets of $H$ and parameters $(28, 9, 2; 1, 4)$

# An algorithm for enumeration of Cayley Neumaier graphs with a spread given by the cosets of a subgroup

**Lemma (Evans, G., Zhao, 2022+)**

*Let $G$ be a finite group and $H < G$ with cosets $Hg_1, \ldots, Hg_n$, with $g_1 = 1$. Then, for any $i \in \{2, \ldots, n\}$,*

$$\mathrm{Stab}_{\mathrm{Aut}\,G}(Hg_i) < \mathrm{Stab}_{\mathrm{Aut}\,G}(H)$$

*holds, where the stabilisers are setwise.*

**Corollary (Evans, G., Zhao, 2022+)**

*Let $G$ be a finite group and $H < G$ with cosets $Hg_1, \ldots, Hg_n$, with $g_1 = 1$. Then, for any $i \in \{2, \ldots, n\}$, each automorphism $\varphi \in \mathrm{Stab}_{\mathrm{Aut}\,G}(Hg_i)$ preserves the partition*

$$\{Hg_2, \ldots, Hg_{i-1}, Hg_{i+1}, \ldots, Hg_n\}.$$

*Moreover, if $g_j = g_i g$, then $\varphi(Hg_j) = Hg_i\varphi(g)$.*

# An algorithm for enumeration of Cayley Neumaier graphs with a spread given by the cosets of a subgroup

**Data:**

a feasible tuple of parameters $(v, k, \lambda; m, s)$ of a Neumaier graph;
a finite group $G$, $|G| = v$;
a subgroup $H < G$, $|H| = s$, with cosets $Hg_1, \ldots, Hg_n$, where $g_1 = 1$;
the automorphism group $\operatorname{Aut} G$;
the stabiliser $\operatorname{Stab}_{\operatorname{Aut} G}(Hg_2)$;

**Result:**

the list of Cayley Neumaier graphs with a spread given by the cosets of the subgroup $H$ with parameters $(v, k, \lambda; m, s)$ over $G$ (each graph is given as $\operatorname{Cay}(G, H^* \cup T)$)

# An algorithm for enumeration of Cayley Neumaier graphs with a spread given by the cosets of a subgroup

for non-equivalent (under $\mathrm{Stab}_{\mathrm{Aut}\,G}(Hg_2)$) $m$-subsets $T_2 \subset Hg_2$
  for all correct $m$-subsets $T_3 \subset Hg_3$
    for all correct $m$-subsets $T_4 \subset Hg_3$
      …
      for all correct $m$-subsets $T_n \subset Hg_n$
        if the necessary and sufficient conditions hold then
          save $\mathrm{Cay}(G, H^* \cup \underbrace{T_2 \cup T_3 \ldots \cup T_n}_{T})$;
        end if;
      end for;
      …
    end for;
  end for;
end for;

# Comments on the algorithm

- feasible tuples of parameters of Neumaier graphs can be computed according to necessary conditions on the existence of Neumaier graphs;
- feasible tuples parameters for strictly Neumaier graphs up to 64 vertices can be found in [7, Table 1];
- the multiplication tables of the groups and the generators of their automorphism groups can be taken from GAP;
- representatives of classes of conjugate subgroups of appropriate order can be taken from GAP;
- having the automorpism group of the group and the representatives of classes of conjugate subgroups, it is possible to compute the list of all non-equivalent subgroups;
- the isomorphism tests for the resulting graphs can be done in SAGE or MAGMA

[7] A. Abiad, W. Castryck, M. De Boeck, J. H. Koolen, S. Zeijlemaker, *An infinite class of Neumaier graphs and non-existence results*, Journal of Combinatorial Theory, Series A Volume 193, January 2023, 105684.
https://doi.org/10.1016/j.jcta.2022.105684

# Numerical data

- all our computational efforts so far are devoted to the following feasible tuples of parameters:

$$(64, 21, 8; 2, 8),$$

$$(64, 28, 12; 3, 8),$$

$$(64, 35, 18; 4, 8),$$

$$(64, 42, 26; 5, 8)$$

$$(64, 49, 36; 6, 8);$$

  these are the parameters of the block graphs of orthogonal arrays OA(8,3), OA(8,4), OA(8,5), OA(8,6) and OA(8,7), respectively; we are interested in both strongly regular graphs (SRGs) and strictly Neumaier graphs (SNGs);
- the first and the fourth tuples are complementary (for SRGs) and correspond to parameters of a Latin square graph and its complement, respectively;

# Numerical data

- the second and the third tuples are complementary (for SRGs) and correspond to parameters of prolific Wallis and Wallis2 constructions;
- the fifth tuple correspond to the parameters of the complement of $8 \times 8$-lattice (there exists a unique strongly regular graph with these parameters);
- there are exactly 267 groups of order 64;
- according to the algorithm, the most difficult groups are those that have many involutions; thus, the most difficult group is $C2 \times C2 \times C2 \times C2 \times C2 \times C2$;
- it is expected to take about 2 months to make computations over $C2 \times C2 \times C2 \times C2 \times C2 \times C2$ (one month has been passed already);
- in other cases, the time needed to execute the program varies from several minutes to several days;
- currently, only 7 groups of order 64 (among 267 such groups) are unfinished; they have numbers 250,261,263,264,265,266 and 267 in GAP

# Findings so far

- at least 6 SRGs and no SNGs with parameters $(64, 21, 8; 2, 8)$;
- at least 40 SRGs and at least 6 SNGs with parameters $(64, 28, 12; 3, 8)$;
- at least 123 SRGs and at least 138 SNGs with parameters $(64, 35, 18; 4, 8)$;
- at least 13 SRGs and a unique SNG with parameters $(64, 42, 26; 5, 8)$; some of the complements of the 13 SRGs have no regular cliques and thus cannot be Latin-square graphs; it is known that, for sufficiently large number of vertices, an SRG with parameters of a Latin square graph is a Latin square graph;
- a unique SRG and no SNGs with parameters $(64, 49, 36; 6, 8)$;
- note that some of these graphs may be Cayley graphs over more than one group $G$ and may be given by more than one subgroup $H$ for the same group $G$.

# Further problems

- generalisation of the obtained examples;
- for the SRGs, we need to make isomorphism tests with known examples; this may be difficult because Wallis and Wallis2 constructions, which give SRGs with parameters $(64, 28, 12; 3, 8)$ and $(64, 35, 18; 4, 8)$, are prolific;
- investigating another feasible tuples of parameters including open tuples of parameters of SRGs from Brouwer's list (the smallest are $(96, 35, 10; 2, 6)$ and $(96, 60, 38; 9, 16)$);
- generalisation of the general construction with use of a partition into completely regular codes of radius 1 instead of a partition into perfect 1-codes;

Thank you for your attention!