Cubic Pancake graphs

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A classical and celebrated result [1] states that two random permutations of a set of n elements almost surely generate either the symmetric or the alternating group of degree n, and similar results hold when more generators are considered. In particular, the symmetric group can always be generated by a transposition together with the n-cycle, or by any set of $n-1$ transpositions.

Our interest lies in describing sets of three prefix reversals that generate the entire symmetric group. The importance of fixed–degree pancake graphs, in particular, cubic pancake graphs as models of networks was shown in [2]. The authors have considered cubic pancake graphs as induced subgraphs of the pancake graph and have identified the combinatorial necessary conditions for a triple of distinct prefix reversals to generate the symmetric group. Using these necessary conditions, six generating sets were obtained by showing that the set can simulate the three generators of the shuffle-exchange permutation network generated by the right and left shuffles, and the transposition of the first two elements.

We use the following approach to get other generating sets, and to describe in group–theoretical terms necessary and sufficient conditions for these sets to generate the symmetric group. Let $H = \langle r_n, r_m, r_k \rangle$, where $n > m > k$ acts on $\Omega = \{1, \ldots, n\}$. Then we consider different cases on m and k such that $H = \mathrm{Sym}_\mathrm{n}$ with utilizing results from group theory about generating sets of Sym_n. In particular, all the generating sets for $k = 2$ and $k = 3$ are characterized. Some other sets are also considered, and structural properties of the corresponding Cayley graphs are discussed. In particular, some results on diameters obtained in [3] are shown.

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References

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