On strictly Neumaier graphs

Sergey Goryainov
Shanghai Jiao Tong University &
Krasovskii Insitute of Mathematics and Mechanics

based on joint work in progress with

Rhys Evans and **Dmitry Panasenko**

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Definitions

A k-regular graph on v vertices is called edge-regular with parameters (v, k, λ) if every pair of non-adjacent vertices has λ common neighbours.

An edge-regular graph with parameters (v, k, λ) is called strongly regular with parameters (v, k, λ, μ) if every pair of distinct non-adjacent vertices has μ common neighbours.

A clique in a regular graph is called m-regular if every vertex that doesn't belong to the clique is adjacent to precisely m vertices from the clique. For an m-regular clique, the number m is called the nexus.

A question by Neumaier

For the clique number $\omega(\Gamma)$ of a strongly regular graph Γ, the Delsarte-Hoffman bound holds:

$$\omega(\Gamma) \leq 1 - \frac{k}{\theta_{\min}},$$

where $\theta_{\min}$ is the smallest eigenvalue of Γ.

A clique in a strongly regular graph is regular if and only if it has $1 - \frac{k}{\theta_{\min}}$ vertices; such a clique is called a Delsarte clique.

In 1981, Neumaier proved [1] that an edge-regular graph which is vertex-transitive, edge-transitive, and has a regular clique is strongly regular.

Neumaier then asked: "**Is it true that every edge-regular graph with a regular clique is strongly regular?**"

[1] A. Neumaier, *Regular Cliques in graphs and Special $1\frac{1}{2}$-designs*, Finite Geometries and Designs, London Mathematical Society Lecture Note Series, 245–259 (1981).

Neumaier graphs

A non-complete edge-regular graph with parameters (v, k, λ) containing an m-regular s-clique is said to be a Neumaier graph with parameters $(v, k, \lambda; m, s)$.

A Neumaier graph that is not strongly regular is said to be a strictly Neumaier graph.

For a Neumaier graph, a spread is a partition of the vertex set into regular cliques.

Outline

1. A construction of strictly Neumaier graphs with 1-regular cliques by Greaves & Koolen and new questions;
2. Four more strictly Neumaier graphs on 24 vertices found in the list of small Cayley-Deza graphs and ‘another’ construction of strictly Neumaier graphs with 1-regular cliques by Greaves & Koolen;
3. A generalisation of Greaves & Koolen’s constructions
4. New strictly Neumaier graphs on 28, 40 and 65 vertices.
5. Determination of the smallest strictly Neumaier graph and a construction of strictly Neumaier graphs with 2^i-regular cliques, for every positive integer i, by Evans, G. & Panasenko;
6. A variation of the Godsil-McKay switching and its application to strictly Neumaier graphs
7. Some directions for further investigation

The first construction of strictly Neumaier graphs

In [2], Greaves and Koolen constructed an infinite family of strictly Neumaier graphs with 1-regular cliques.

For positive integers ℓ, m and an odd prime power q, consider the group $G_{\ell,m,q} := \mathbb{Z}_\ell \oplus \mathbb{Z}_2^m \oplus \mathbb{F}_q$. Put

$$S_0 := \{(x, y, 0) \mid x \in \mathbb{Z}_\ell, y \in \mathbb{Z}_2^m, (x, y) \neq (0, 0)\}$$

Let $\pi : \mathbb{Z}_2^m \setminus \{0\} \to \{0, \ldots, 2m - 2\}$ be a bijection and ρ be a primitive element of $\mathbb{F}_q$.

For each $y \in \mathbb{Z}_2^m \setminus \{0\}$, define

$$S_{y,\pi} := \{(0, y, \rho^j) \mid \pi(y) \equiv j \ (mod\ 2^m - 1)\}$$

Consider the parametrised Cayley graph $Cay(G_{\ell,m,q}, S(\pi))$, where

$$S(\pi) := S_0 \cup \bigcup_{y \in \mathbb{Z}_2^m \setminus \{0\}} S_{y,\pi}$$

[2] G. R. W. Greaves, J. H. Koolen, *Edge-regular graphs with regular cliques*, Europ. J. Combin., 71, 194–201 (2018).

The first construction of strictly Neumaier graphs

Let $q = 2nr + 1$ for some positive integer r. For each $i \in \{0, \dots, n-1\}$, define the cyclotomic class

$$C_q^n(i) := \{\rho^{nj+i} \mid j \in 0, \dots, 2r-1\}.$$

For $a, b \in \{0, \dots, n-1\}$, define the cyclotomic number

$$c_q^n(a, b) := |C_q^n(a) + 1 \cap C_q^n(b)|.$$

Put $c := c_q^n(a, b)$ and $\ell := (1 + c)/2$.

Theorem ([2, Theorem 3.6, Corollary 4.4])

Let $q \equiv 1 \pmod 6$, c be odd and $\pi : \mathbb{Z}_2^2 \setminus \{0\} \to \{0, 1, 2\}$ be a bijection. Then $Cay(G_{\ell,2,q}, S(\pi))$ is a strictly Neumaier graph with parameters $(4\ell q, 4\ell - 2 + q, 4\ell - 2; 1, 4\ell)$.

[2] G. R. W. Greaves, J. H. Koolen, *Edge-regular graphs with regular cliques*, Europ. J. Combin., 71, 194–201 (2018).

Notes on the first construction

- Set $q := 7^a$, where $a \not\equiv 0 \ (mod\ 3)$. Then $Cay(G_{\ell,2,q}, S(\pi))$ is a strictly Neumaier graph with parameters

$$(4\ell q, 4\ell - 2 + q, 4\ell - 2; 1, 4\ell).$$

  In particular, if $a = 1$, then we have a strictly Neumaier graph with parameters $(28, 9, 2; 1, 4)$. This graph is the smallest example from [2].
- $Cay(G_{\ell,2,q}, S(\pi))$ has a spread of size q given by the cosets of the subgroup $\{(x, y, 0) \mid x \in \mathbb{Z}_\ell, y \in \mathbb{Z}_2^m\}$.

[2] G. R. W. Greaves, J. H. Koolen, *Edge-regular graphs with regular cliques*, Europ. J. Combin., 71, 194–201 (2018).

Four strictly Neumaier graphs on 24 vertices

Gavrilyuk and Goryainov then searched for examples in a collection of known Cayley-Deza graphs [3] and found four more strictly Neumaier graphs with parameters $(24, 8, 2; 1, 4)$.

In [4], Greaves and Koolen found ‘another’ infinite family of strictly Neumaier graphs, which contains one of the four graphs on 24 vertices.

[3] S. V. Goryainov, L. V. Shalaginov, *Cayley-Deza graphs with fewer than 60 vertices*, Siberian Electronic Mathematical Reports, 11, 268–310 (2014) (in Russian).

[4] G. R. W. Greaves, J. H. Koolen, *Another construction of edge-regular graphs with regular cliques*, Discrete Mathematics, to appear, `https://doi.org/10.1016/j.disc.2018.09.032`

Antipodal distance-regular graphs

A graph Γ of diameter d is called distance-regular if, for any two vertices $x, y \in V(\Gamma)$, the number of vertices at distance i from x and distance j from y depends only on i, j, and the distance from x to y. It is clear that distance regular graphs are edge-regular.

A distance-regular graph Γ of diameter d is called a-antipodal if the relation of being at distance d or distance 0 is an equivalence relation on the vertices of Γ with equivalence classes of size a.

The second construction of strictly Neumaier graphs

Let Γ be an a-antipodal distance-regular graph of diameter 3 with edge-regular parameters (v, k, λ) such that a is a proper divisor of $\lambda + 2$.

Put $t = \frac{\lambda+2}{a}$ and take t disjoint copies $\Gamma^{(1)}, \ldots, \Gamma^{(t)}$ of Γ.

For every antipodal class H in Γ, take the corresponding antipodal classes $H^{(1)}, \ldots, H^{(t)}$ in $\Gamma^{(1)}, \ldots, \Gamma^{(t)}$, respectively, and connect any two vertices from $H^{(1)} \cup \ldots \cup H^{(t)}$ to form a 1-regular clique of size at.

Denote by $F_t(\Gamma)$ the resulting graph.

Theorem ([4])

The graph $F_t(\Gamma)$ is a strictly Neumaier graph having parameters $(tv, k + at - 1, \lambda; 1, at)$ and containing a spread.

[4] G. R. W. Greaves, J. H. Koolen, *Another construction of edge-regular graphs with regular cliques*, Discrete Mathematics, to appear, `https://doi.org/10.1016/j.disc.2018.09.032`

Notes on the second construction

- In particular, if Γ is the icosahedron, then $a = 2$, $\lambda = 2$, $t = 2$ and $F_2(\Gamma)$ is one of the four strictly Neumaier graphs with parameters $(24, 8, 2; 1, 4)$ found in [3].
- The other three graphs can be obtained in a similar way by choosing an appropriate matching of the antipodal classes in the two copies of the icosahedrons.

[3] S. V. Goryainov, L. V. Shalaginov, *Cayley-Deza graphs with fewer than 60 vertices*, Siberian Electronic Mathematical Reports, 11, 268–310 (2014) (in Russian).

A generalisation of the first and the second costructions

Let $\Gamma^{(1)}, \ldots, \Gamma^{(t)}$ edge-regular graphs with parameters (v, k, λ) that admit a partition into perfect 1-codes of size a, where a is a proper divisor of $\lambda + 2$ and $t = \frac{\lambda+2}{a}$;

For any $j \in \{1, \ldots, t\}$, let $H_1^{(j)}, \ldots, H_{\frac{v}{a}}^{(j)}$ denote the perfect 1-codes that partition the vertex set of $\Gamma^{(j)}$.

Let $\Pi = (\pi_2, \ldots, \pi_t)$ be a $(t-1)$-tuple of permutations from $Sym(\{1, \ldots, \frac{v}{a}\})$.

1. Take the disjoint union of the graphs $\Gamma^{(1)}, \ldots, \Gamma^{(t)}$.
2. For any $i \in \{1, \ldots, \frac{v}{a}\}$, connect any two vertices from $H_i^{(1)} \cup H_{\pi_2(i)}^{(2)} \cup \ldots \cup H_{\pi_t(i)}^{(t)}$ to form a 1-regular clique of size at.
3. Denote by $F_\Pi(\Gamma^{(1)}, \ldots, \Gamma^{(t)})$ the resulting graph, which is a strictly Neumaier graph whose vertex set has been partitioned into 1-regular cliques.

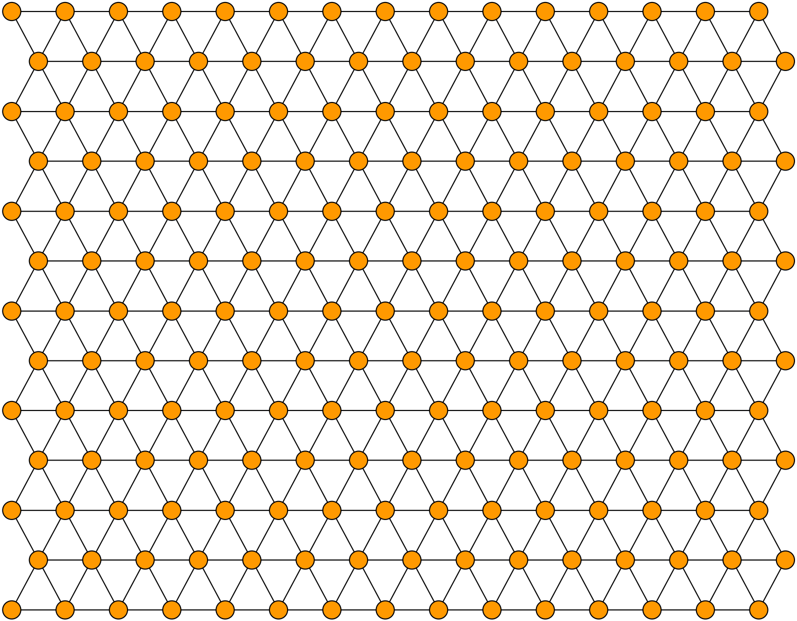
Notes on the generalisation

- Non-isomorphic Taylor graphs with the same parameters give many new examples in the case $t \geq 2$.
- The four strictly Neumaier graphs on 24 vertices from [3] are given by a pair of icosahedrons, and the only difference between them is the choice of the permutation that matches the antipodal classes.
- The generalised construction covers both constructions from [2] and [4] (the cases $t = 1$ and $t \geq 2$, respectively).
- For $t = 1$ we can construct three new strictly Neumaier graphs with parameters $(28, 9, 2; 1, 4)$, $(40, 12, 2; 1, 4)$ and $(65, 16, 3; 1, 5)$.

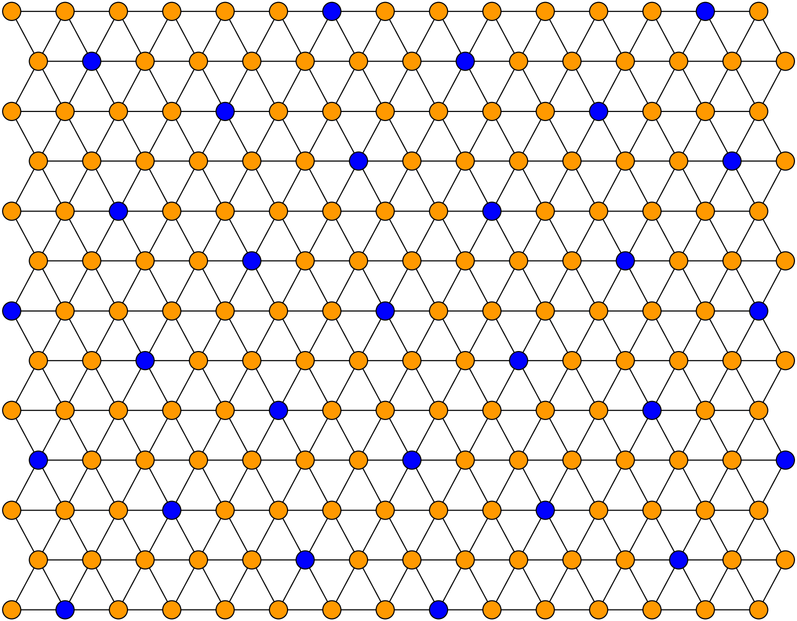
[2] G. R. W. Greaves, J. H. Koolen, *Edge-regular graphs with regular cliques*, Europ. J. Combin., 71, 194–201 (2018).
[3] S. V. Goryainov, L. V. Shalaginov, *Cayley-Deza graphs with fewer than 60 vertices*, Sibirskie Èlektronnye Matematicheskie Izvestiya, 11, 268–310 (2014).

[4] G. R. W. Greaves, J. H. Koolen, *Another construction of edge-regular graphs with regular cliques*, Discrete Mathematics, to appear,

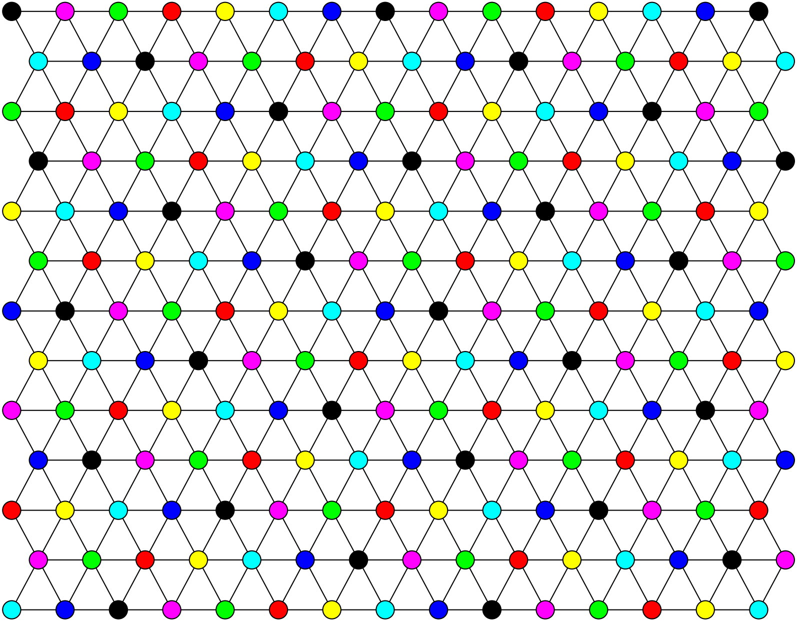
Triangular grid



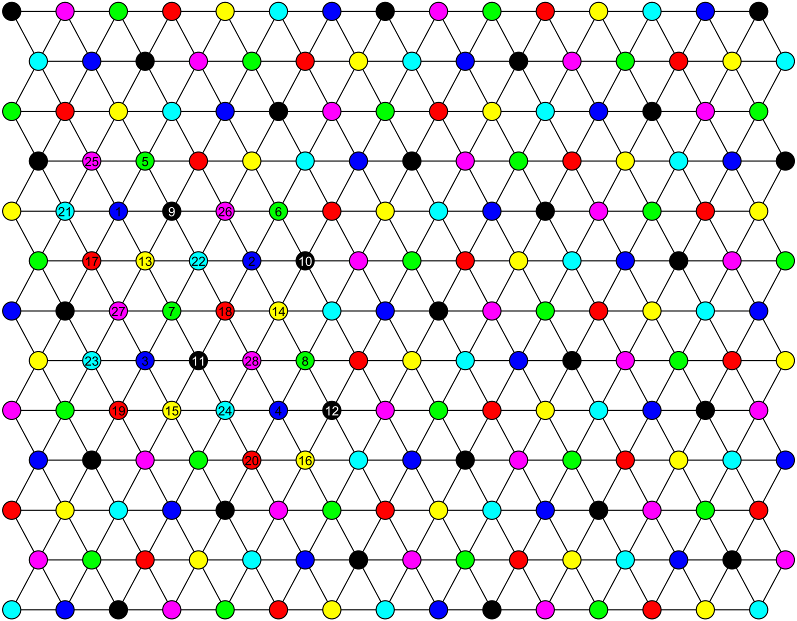
Perfect 1-code



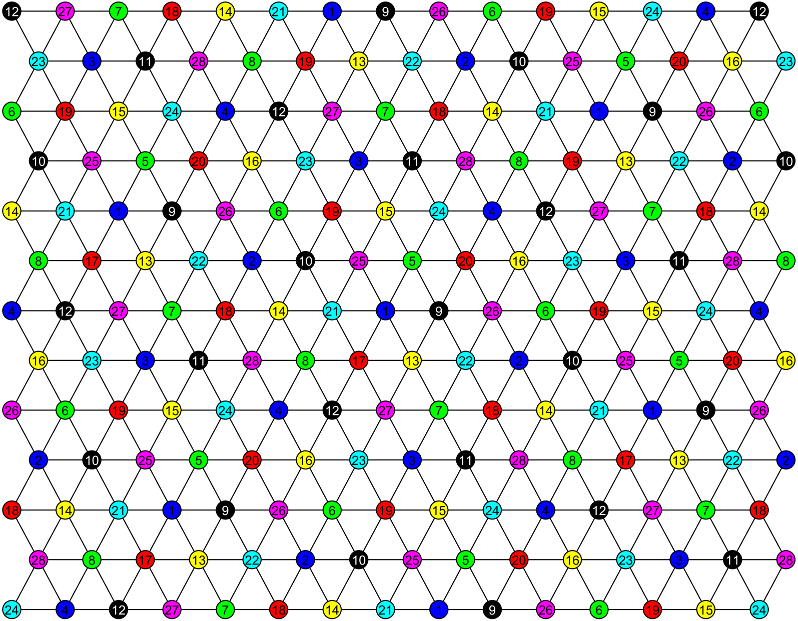
Partition into perfect 1-codes



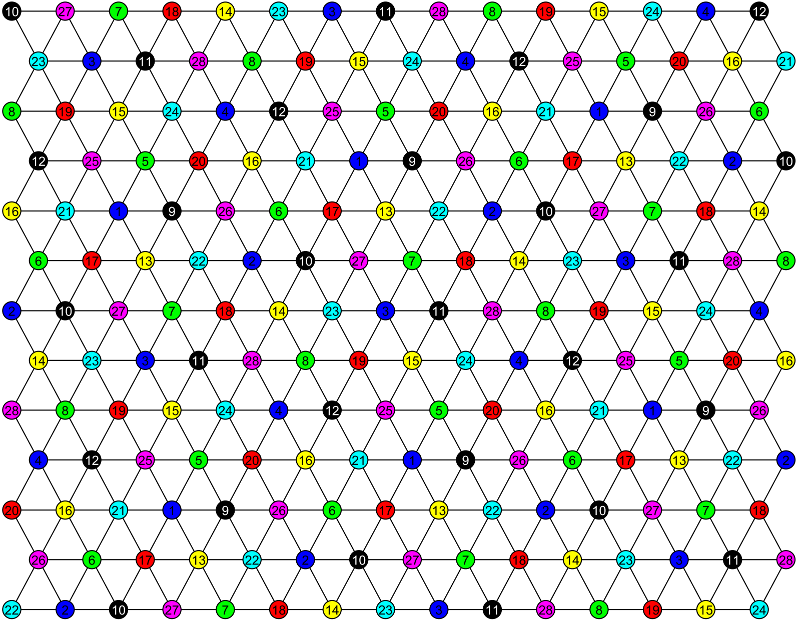
Four balls



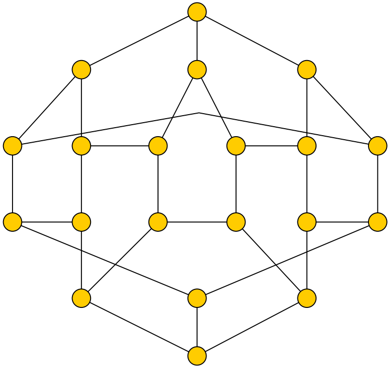
Quotient 1: isomorphic to the known (28,9,2;1,4)-graph



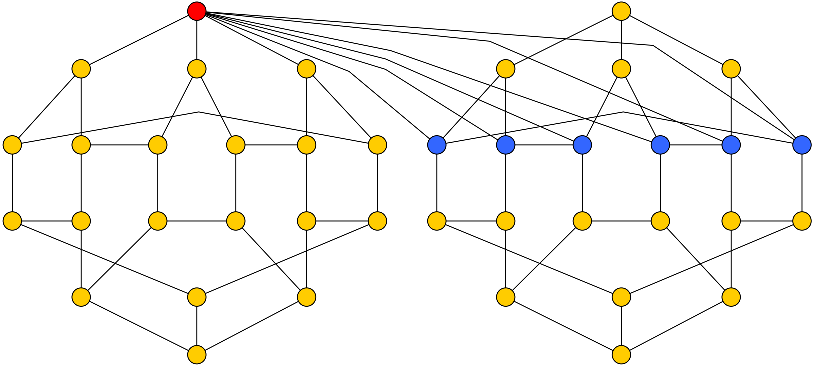
Quotient 2: new strictly Neumaier graph, (28,9,2;1,4)



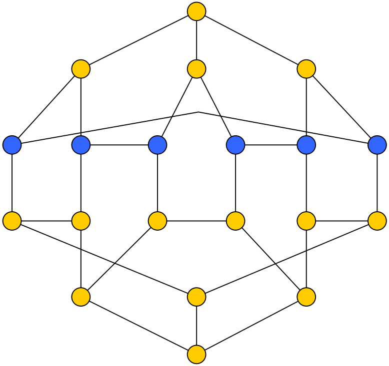
A pair of disjoint dodecahedrons

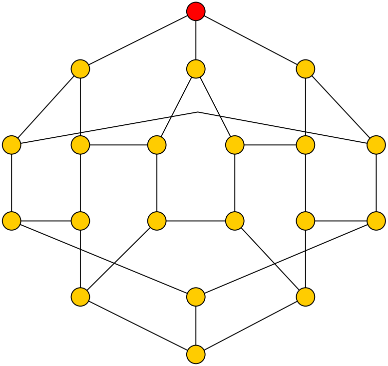


Additional 6 neighbours for every vertex

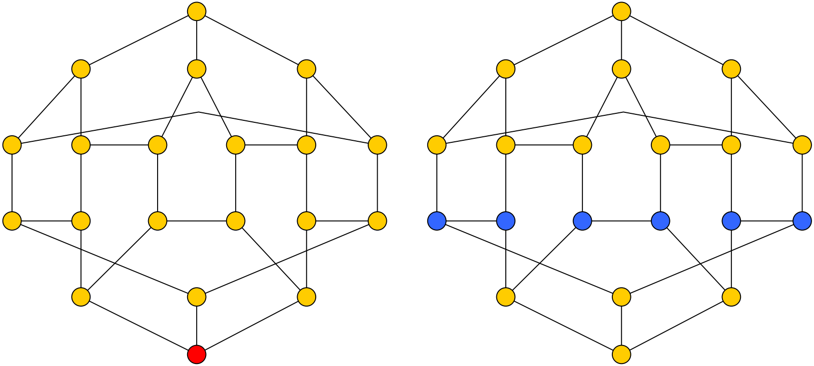


Additional 6 neighbours for every vertex

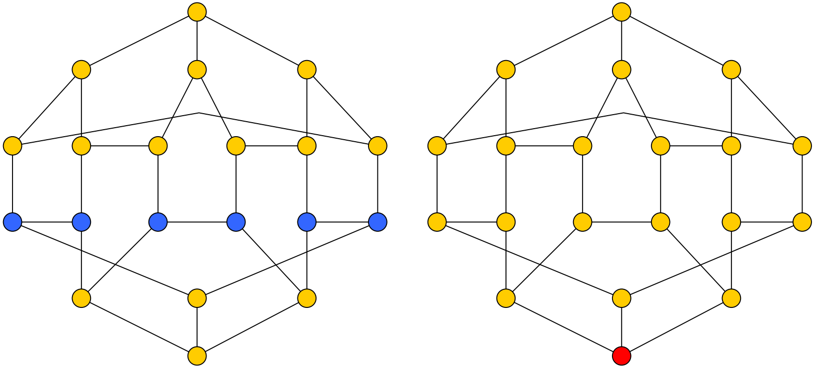




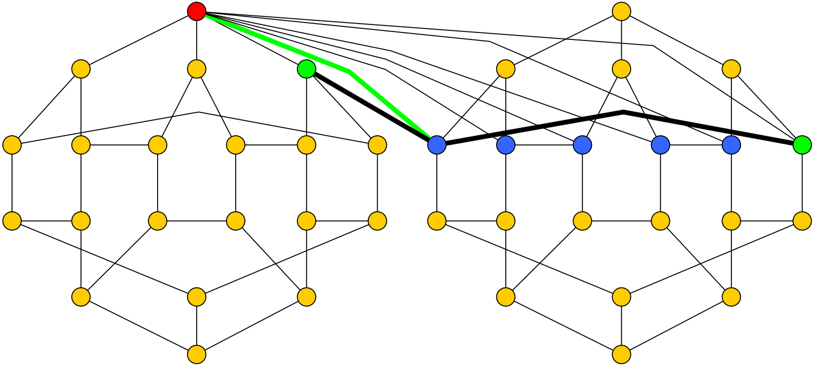
Additional 6 neighbours for every vertex



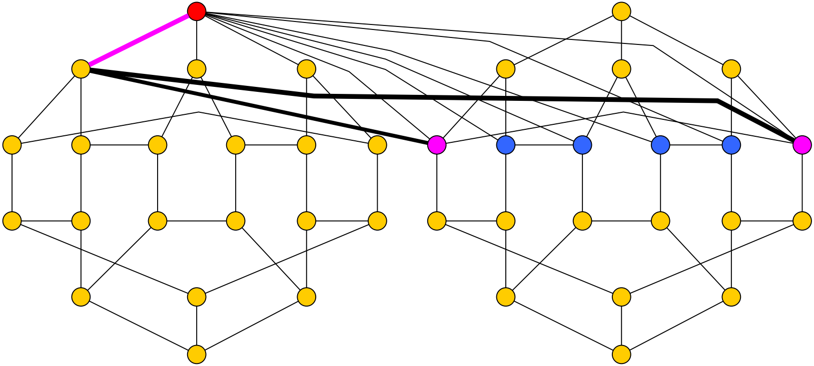
Additional 6 neighbours for every vertex



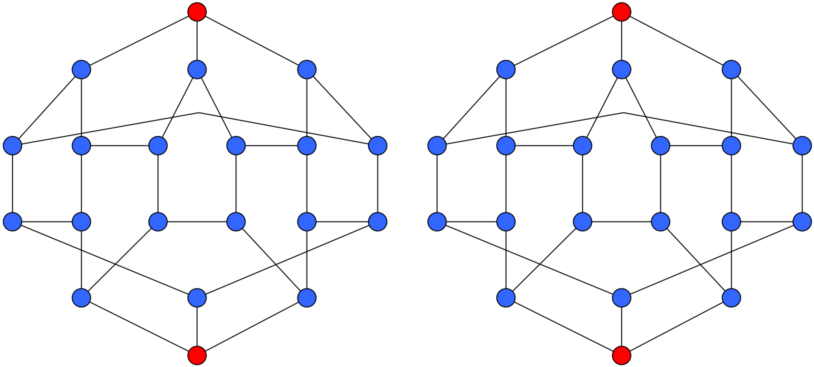
The first type of adjacent vertices



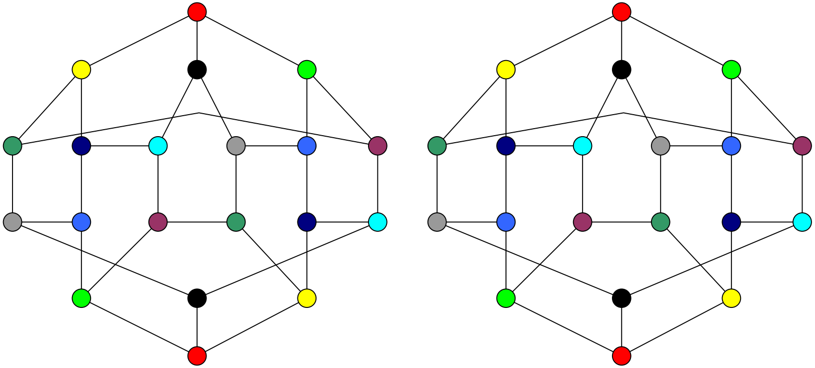
The second type of adjacent vertices



Perfect 1-code in the (40,9,2)-edge-regular graph



Partition into perfect codes gives a (40,12,2;1,4)-graph



New strictly Neumaier graph, (65,16,3;1,5)

The element 2 has order 12 modulo 65.

Consider the circulant $Cay(\mathbb{Z}_{65}, \{1, 2, 2^2, \ldots, 2^{11}\})$, which is edge-regular with parameters $(65, 12, 3)$.

The cosets of the subgroup of order 5 form a partition into perfect 1-codes.

Finally, we obtain a strictly Neumaier graph with parameters $(65, 16, 3; 1, 5)$.

Strictly Neumaier graphs with 2^i-regular cliques

In [5], Evans, Goryainov and Panasenko found a strictly Neumaier graph containing a 2^i-regular clique for every positive integer i.

The smallest graph in this family has parameters (16,9,4;2,4).

It was also proved that this graph on 16 vertices is the smallest strictly Neumaier graph (w.r.t the number of vertices).

[5] R. J. Evans, S. V. Goryainov, D. I. Panasenko, The smallest strictly Neumaier graph and its generalisations, The Electronic Journal of Combinatorics, 26(2) (2019), P2.29.

Affine polar graph

Let V be a $(2e)$-dimensional vector space over a finite field $\mathbb{F}_q$, where $e \geq 2$ and q is a prime power, provided with the hyperbolic quadratic form $Q(x) = x_1x_2 + x_3x_4 + \ldots + x_{2e-1}x_{2e}$.

The set Q^+ of zeroes of Q is called the hyperbolic quadric, where e is the maximal dimension of a subspace in Q^+. A generator of Q^+ is a subspace of maximal dimension e in Q^+.

Denote by $VO^+(2e, q)$ the graph on V with two vectors x, y being adjacent iff $Q(x - y) = 0$.

The graph $VO^+(2e, q)$ is known to be a vertex transitive strongly regular graph with parameters

$$v = q^{2e}, k = (q^{e-1} + 1)(q^e - 1),$$

$$\lambda = q(q^{e-2} + 1)(q^{e-1} - 1) + q - 2, \mu = q^{e-1}(q^{e-1} + 1).$$

Affine polar graph

Note that $VO^+(2e,q)$ is isomorphic to the graph defined on the set of all $(2\times e)$-matrices over $\mathbb{F}_q$

$$\{\begin{pmatrix} x_1 & x_3 & \dots & x_{2e-1} \\ x_2 & x_4 & \dots & x_{2e} \end{pmatrix}\},$$

where two matrices are adjacent iff the scalar product of the first and the second rows of their difference is equal to 0.

A spread in $VO^+(2e,q)$ is a set of q^e disjoint maximal cliques that correspond to all cosets of a generator.

The smallest strictly Neumaier graph

Put $e = 2$ and $q = 2$, and consider the 1-dimensional subspace

$$W = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}.$$

The subspace W is contained in the two generators

$$W_1 = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix} \text{ and } W_2 = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}.$$

Take the vector

$$v = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and consider the cosets

$$v + W_1 = \begin{pmatrix} * & * \\ 1 & 0 \end{pmatrix}, \quad v + W_2 = \begin{pmatrix} * & 0 \\ 1 & * \end{pmatrix},$$

whose intersection is

$$v + W = \begin{pmatrix} * & 0 \\ 1 & 0 \end{pmatrix}.$$

The smallest strictly Neumaier graph

The switching edges between the cliques W_1, $v + W_1$ gives a graph isomorphic to the complement of the Shrikhande graph.

The switching edges between the cliques W_1, $v + W_1$ and then between the cliques W_2, $v + W_2$ gives the smallest strictly Neumaier graph, which is vertex-transitive, has parameters (16,9,4;2,4) and contains a spread.

A generalisation of the switching

This idea also works in the general case $e \geq 2$.

Take the $(e-1)$-dimensional subspace

$$W = \left(\begin{array}{ccc|cc} * & \dots & * & * & 0 \\ 0 & \dots & 0 & 0 & 0 \end{array} \right),$$

The subspace W is contained in the two generators

$$W_1 = \left(\begin{array}{ccc|cc} * & \dots & * & * & * \\ 0 & \dots & 0 & 0 & 0 \end{array} \right) \text{ and } W_2 = \left(\begin{array}{ccc|cc} * & \dots & * & * & 0 \\ 0 & \dots & 0 & 0 & * \end{array} \right).$$

Take the vector

$$v = \left(\begin{array}{ccc|cc} 0 & \dots & 0 & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 \end{array} \right)$$

and consider the cosets

$$v+W_1 = \left(\begin{array}{ccc|cc} * & \dots & * & * & * \\ 0 & \dots & 0 & 1 & 0 \end{array} \right), \quad v+W_2 = \left(\begin{array}{ccc|cc} * & \dots & * & * & 0 \\ 0 & \dots & 0 & 1 & * \end{array} \right),$$

whose intersection is

$$v + W = \left(\begin{array}{ccc|cc} * & \dots & * & * & 0 \\ 0 & \dots & 0 & 1 & 0 \end{array} \right).$$

A generalisation of the switching

The switching edges between the cliques W_1, $v + W_1$ gives a strongly regular graph which has parameters the same as the affine polar graph $VO^+(2e, 2)$.

The switching edges between the cliques W_1, $v + W_1$ and then between the cliques W_2, $v + W_2$ gives a strictly Neumaier graph, which is not vertex-transitive and contains a 2^{e-1}-regular clique of size 2^e.

A variation of the Godsil-McKay switching

Let Γ be a graph whose vertex set is partitioned as $C_1 \cup C_2 \cup D$. Assume that $|C_1| = |C_2|$ and that the induced subgraphs on C_1, C_2, and $C_1 \cup C_2$ are regular, where the degrees in the induced subgraphs on C_1 and C_2 are the same. Suppose that all $x \in D$ satisfy one of the following

1. $|\Gamma(x) \cap C_1| = |\Gamma(x) \cap C_2|$, or
2. $\Gamma(x) \cap (C_1 \cup C_2) \in \{C_1, C_2\}$.

Construct a graph Γ' from Γ by modifying the edges between $C1 \cup C2$ and D as follows:

$$\Gamma'(x) \cap (C_1 \cup C_2) := \begin{cases} C_1, & \text{if } \Gamma(x) \cap (C_1 \cup c_2) = C_2; \\ C_2, & \text{if } \Gamma(x) \cap (C_1 \cup c_2) = C_1; \\ \Gamma(x) \cap (C_1 \cup C_2), & \text{otherwise.} \end{cases}$$

[6] W. Wang, L. Qiu, Y. Hu, Cospectral graphs, GM-switching and regular rational orthogonal matrices of level p, Linear Algebra and its Applications, Volume 563, 15 (2019), 154–177. [7] F. Ihringer, A. Munemasa, New Strongly Regular Graphs from Finite Geometries via Switching, `https://arxiv.org/pdf/1904.03680.pdf`

Applications of the variation of GM-switching

- Twisting of cliques in the generalised construction in the case $t \geq 2$;
- Switching edges between two regular cliques of $VO^{+}(2e, 2)$ from the same spread

A question from the book "Distance-regular graphs"

The complement of a $n \times (n+1)$-lattice is an edge-regular graph whose parameters k and λ satisfy the equality

$$\lambda = k + 1 - \sqrt{4k+1}.$$

In [8,p.13], the following problem has been formulated:
"**Is every edge-regular graph with parameters** (n, k, λ) **satisfying**

$$\lambda > k + 1 - \sqrt{4k+1}.$$

necessarily strongly regular?"
For the smallest strictly Neumaier graph (edge-regular with parameters $(16, 9, 4)$) we have

$$4 > 10 - \sqrt{37}.$$

However, all other graphs from the infinite family of strictly Neumaier graphs do not satisfy this inequality.

[8] A. E. Brouwer, A. M. Cohen, and A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag, Berlin (1989).

Thank you for your attention!