Thin divisible designs graphs: an interplay between fixed-point free involutions of (v, k, λ)-graphs and symmetric weighing matrices

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1. Introduction to divisible design graphs

Group divisible designs

A group divisible design with parameters $(v, k, \lambda_1, \lambda_2, m, n)$ is an incidence structure on v points, with blocks of size k, having the property that the point set can be partitioned into m classes of such n such that any pair of distinct points from the same class occurs together in exactly λ_1 blocks and any pair of points from different classes occurs together in exactly λ_2 blocks.

Clearly, group divisible designs are a generalisation of (v, k, λ)-designs.

A group divisible design D with parameters $(v, k, \lambda_1, \lambda_2, m, n)$ is said to have dual property if the dual incidence structure D' is a group divisible design with the same parameters.

Clearly, group divisible designs with dual property are a generalisation of symmetric (v, k, λ)-designs.

[B77] R.C. Bose, *Symmetric group divisible designs with the dual property*, Journal of Statistical Planning and Inference, Volume 1, Issue 1, February 1977, Pages 87–101.

(v, k, λ)-graphs and (v, k, λ)-designs

An undirected k-regular graph X on v vertices is called a strongly regular graph with parameters (v, k, λ, μ) if any two adjacent vertices of X have exactly λ common neighbours and any two distinct non-adjacent vertices of X have exactly μ common neighbours.

A strongly regular graph with parameters (v, k, λ, μ) such that $\lambda = \mu$ is called a (v, k, λ)-graph [R71].

The motivation was that this gives an interplay between (strongly regular) graphs and (symmetric) designs. This connection can be useful for both parts. On one hand, the adjacency matrix of a (v, k, λ)-graph can be viewed as the incidence matrix of a (v, k, λ)-design. On the other hand, for example the easiest construction of a $(16, 6, 2)$ biplane is via the $(16, 6, 2)$-graph which is just the line graph of $K_{4,4}$.

[R71] A. Rudvalis, (v, k, λ)-*Graphs and polarities of* (v, k, λ)-*designs*, Mathematische Zeitschrift, 120 (1971) 224–230.

Divisible design graphs

A similar motivation holds for divisible design graphs, which were introduced in 2011 in [HKM11]. A k-regular graph on v vertices is called a divisible design graph (DDG for short) with parameters $(v, k, \lambda_1, \lambda_2, m, n)$ if its vertex set can be partitioned into m classes of size n such that any two vertices from the same class have exactly λ_1 common neighbours and any two vertices from different classes have exactly λ_2 common neighbours.

The adjacency matrix of a divisible design graph can be viewed as the incidence matrix of a group divisible design.

Thus, studying divisible design graphs enriches the theory of group divisible designs and vice versa.

Note that a DDG with $m = 1$, $n = 1$, or $\lambda_1 = \lambda_2$ is a (v, k, λ)-graph. In this case, we call the DDG improper, otherwise it is called proper.

[HKM11] W. H. Haemers, H. Kharaghani, M. A. Meulenberg, *Divisible design graphs*, Journal of Combinatorial Theory, Series A, 118(3) (2011) 978–992.

Spectrum of a DDG

Proposition 1 ([HKM11, Lemma 2.1])

The eigenvalues of the adjacency matrix of a DDG with parameters $(v, k, \lambda_1, \lambda_2, m, n)$ *are*

$$\left\{k^1, \left(\sqrt{k-\lambda_1}\right)^{f_1}, \left(-\sqrt{k-\lambda_1}\right)^{f_2}, \left(\sqrt{k^2-\lambda_2 v}\right)^{g_1}, \left(-\sqrt{k^2-\lambda_2 v}\right)^{g_2}\right\},$$

where $f_1 + f_2 = m(n-1)$ *and* $g_1 + g_2 = m - 1$.

In general, the multiplicities f_1, f_2, g_1 and g_2 are not determined by the parameters, but if we know one, we know them all because

$$f_1 + f_2 = m(n-1),$$

$$g_1 + g_2 = m - 1,$$

$$\operatorname{trace} A = 0 = k + (f_1 - f_2)\sqrt{k-\lambda_1} + (g_1 - g_2)\sqrt{k^2 - \lambda_2 v}.$$

The canonical partition of a DDG

The partition from the definition is unique and gives a partition of the adjacency matrix:

$$A = \begin{bmatrix} A_{1,1} & \cdots & A_{1,m} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,m} \end{bmatrix}.$$

For a DDG, this partition is called the canonical partition.

Theorem 1 ([HKM11, Theorem 3.1])

The canonical partition of the adjacency matrix of a proper DDG is equitable, and the quotient matrix R satisfies

$$R^2 = RR^T = (k^2 - \lambda_2 v)I_m + \lambda_2 n J_m,$$

and the eigenvalues of R are

$$\left\{ k^1, \left(\sqrt{k^2 - \lambda_2 v}\right)^{g_1}, \left(-\sqrt{k^2 - \lambda_2 v}\right)^{g_2} \right\}.$$

Almost proper DDGs and the state of art

If for an improper DDG (that is, a (v, k, λ)-graph) the partition from the definition is equitable and $m, n > 1$, we call the improper DDG almost proper. The justification for this new term is that although the DDG is not proper, it behaves much as a proper DDG.

Currently, there are tens constructions producing infinitely many proper DDGs known. These constructions make use of many combinatorial and algebraic objects: finite geometries, Hadamard matrices, weighing matrices, designs, Cayley graphs, block matrices, strongly regular graphs, distance-regular graphs and so on.

Also, a number of characterisations of divisible design graphs are known.

Within this talk, we will present our recent developments on almost proper DDGs.

2. Thin divisible design graphs

Thin divisible design graphs

A divisible design graph with $n = 2$ (that is, when the classes of the canonical partition have size 2) is called thin.

For a thin DDG, there are only the following four possible blocks in the canonical partition of the adjacency matrix:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

In the case of thin DDGs, the quotient matrix R has a natural partner Q of the same size. If $A_{ij} = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ is a block of A, then the corresponding element of Q is defined by $(Q)_{i,j} = a - b$. Since $(R)_{i,j} = a + b$, we have $Q \equiv R \pmod 2$. The entries of R are 0, 1 or 2, and those of Q are -1, 0 and 1. Note that R and Q are symmetric matrices.

[CH14] D. Crnković, W. H. Haemers, *Walk-regular divisible design graphs*, Designs, Codes and Cryptography, Volume 72, 165–175, 2014.

Matrix Q as a weighing matrix

Theorem 2 ([CH14, Theorem 4.3(b)])

The eigenvalues of Q are $\theta_1 = \sqrt{k - \lambda_1}$ and $\theta_2 = -\theta_1$ with multiplicities f_1 and $f_2 = m - f_1$, respectively.

Note that the spectrum of A is the union of the spectra of R and Q.

Moreover, $Q^2 = (k - \lambda_1)I$ holds. Thus, Q is a symmetric weighing matrix of weight $k - \lambda_1$ whose diagonal entries are equal to 0 and -1.

Recall that a weighing matrix of order v and weight w is a matrix W with entries from the set $\{0, 1, -1\}$ such that $W \cdot W^T = wI_v$, where W^T is the transpose of W and I_v is the identity matrix of order v.

Note that if all the diagonal blocks of the canonical partition of the matrix A are zero blocks, then the matrix Q has zero diagonal and can be viewed as the adjacency matrix of a simple signed graph having exactly two distinct eigenvalues. The problem of classification of such signed graphs is widely open in the theory of signed graphs.

Thus, thin divisible design graphs are important as they give rise to weighing matrices and signed graphs with exactly two eigenvalues.

Signings of simple graphs

In [H19], Hao Huang proved the sensitivity conjecture. A key step of his proof is finding a signing of the edges of a hypercube H_n such that the adjacency matrix of the resulting signed graphs is a weighing matrix (has exactly two distinct eigenvalues).

Thus, the following problem is of interest.

Problem 1

Given a simple graph G, does there exist a signing σ of G such that the signed graph (G, σ) has exactly two distinct eigenvalues.

Note that if G is a complete graph, then Problem 1 is equivalent to the problem of constructing a symmetric conference matrix.

Further, we discuss a construction of thin divisible design graphs based on symmetric weighing matrices.

[H19] H. Huang, *Induced subgraphs of hypercubes and a proof of the sensitivity conjecture*, Annals of Mathematics 190, 3 (2019), 949–955.

Thin DDGs from weighing matrices

Theorem 3 ([CH14, Theorem 4.4])

Let Q be a symmetric weighing matrix of order m and weight w satisfying $(Q)_{i,i} \neq 1$ $(i = 1, \dots, m)$. Let R be a symmetric $(0, 1, 2)$-matrix, satisfying $R \equiv Q \pmod 2$, $(R)_{i,i} \neq 2$ $(i = 1, \dots, m)$, and $R^2 = \alpha I + \beta J$ for some $\alpha, \beta \in \mathbb{R}$. Then

$$A = \frac{1}{2} \begin{bmatrix} R + Q & R - Q \\ R - Q & R + Q \end{bmatrix}$$

is the adjacency matrix of a thin DDG with quotient matrix R, partner Q and parameters

$$(v = 2m, k = \sqrt{\alpha + m\beta}, \lambda_1 = k - w, \lambda_2 = \beta/2, m, 2).$$

Further we discuss how to satisfy the conditions of Theorem 3 with use of regular symmetric Hadamard matrices with constant diagonal.

[CH14] D. Crnković, W. H. Haemers, *Walk-regular divisible design graphs*, Designs, Codes and Cryptography, Volume 72, 165–175, 2014.

3. Hadamard matrices

Regular symmetric Hadamard matrices with constant diagonal (I)

An Hadamard matrix is a square matrix whose entries are either $+1$ or -1 and whose rows are mutually orthogonal.

An Hadamard matrix H is called symmetric if $H = H^T$.

An Hadamard matrix H is called regular when all row sums are equal. If J denotes the all-1 matrix of order n, then all row sums are equal to a if and only if $HJ = aJ$. (It follows that $JH = aJ$ and $a^2 = n$.)

An Hadamard matrix $H = (h_{ij})$ has constant diagonal when $h_{ii} = e$ for all i and some $e \in \{\pm 1\}$.

Abbreviate the phrase 'regular symmetric Hadamard matrix with constant diagonal' with RSHCD.

Let H be a RSHCD with parameters n, a, e. Then $a^2 = n$ so that $a = \pm\sqrt{n}$. The matrix $-H$ is a RSHCD with parameters $n, -a, -e$, so that there are the two essentially distinct cases $ae > 0$ and $ae < 0$. Put $ae = \varepsilon\sqrt{n}$ with $\varepsilon \in \{\pm 1\}$, and call H of type ε. If $n > 1$, then $4|n$, so $2|a$, say $a = 2u$.

Regular symmetric Hadamard matrices with constant diagonal (II)

Then $A = \frac{1}{2}(J - eH)$ is the adjacency matrix of a strongly regular graph (degenerate for $(n, \varepsilon) = (4, -1)$) with parameters

$$v = 4u^2, \quad k = 2u^2 - \varepsilon u, \quad \lambda = \mu = u^2 - \varepsilon u.$$

And $J - I - A = \frac{1}{2}(J + eH - 2I)$ is the adjacency matrix of the complementary strongly regular graph with parameters

$$v = 4u^2, \quad k = 2u^2 + \varepsilon u - 1, \quad \lambda = u^2 + \varepsilon u - 2 \quad \mu = u^2 + \varepsilon u.$$

Conversely, graphs with these parameters yield RSHCDs.

Note that A is the incidence matrix of a symmetric (v, k, λ)-design with parameters $(4u^2, 2u^2 \pm u, u^2 \pm u)$ and zero diagonal. Designs with these parameters are known as Menon designs.

Overview of known constructions of RSHCDs (I)

Let S be the set of pairs (n, ε) for which an RSHCD of order n and type ε exists. In the monograph on strongly regular graphs by Brouwer and Van Maldeghem (2022), the following constructions of RSHCDs were surveyed. First, there exists one recursive construction, namely, the Kronecker product

$$(m, \delta), (n, \varepsilon) \in S \Rightarrow (mn, \delta\varepsilon) \in S.$$

Second, there are the following ten direct constructions:

- $(4, \pm 1), (36, \pm 1) \in S$.
- If there exists a Hadamard matrix of order m, then $(m^2, 1) \in S$.
- If both $a-1$ and $a+1$ are odd prime powers, and $4 \mid a$, then $(a^2, 1) \in S$.
- If $a+1$ is a prime power, and there exists a symmetric conference matrix of order a, then $(a^2, 1) \in S$.
- If there is a set of $t-2$ mutually orthogonal latin squares of order $2t$, then $(4t^2, 1) \in S$.

Overview of known constructions of RSHCDs (II)

- Suppose we have a Steiner system $S(2, K, V)$ with $V = K(2K-1)$. If we form the block graph, and add an isolated point, we get a graph in the switching class of a regular two-graph. The corresponding Hadamard matrix is symmetric with constant diagonal, but not regular. If this Steiner system is invariant under a regular abelian group of automorphisms (which necessarily has orbits on the blocks of sizes V, V, and $2K-1$), then by switching with respect to a block orbit of size V we obtain a strongly regular graph with parameters

$$v = 4K^2, \quad k = K(2K-1), \quad \lambda = \mu = K(K-1)$$

  showing that $(4K^2, 1) \in S$. Steiner systems $S(2, K, K(2K-1))$ are known for $K = 3, 5, 6, 7$ or 2^t, but only for $K = 2, 3, 5, 7$ are systems known that have a regular abelian group of automorphisms. Thus, we find $(196, 1) \in S$. The required switching set also exists when the design is resolvable: take the union of K parallel classes. Resolvable designs are known for $K = 3$ or 2^t.

Overview of known constructions of RSHCDs (III)

- $(100, -1) \in S$.
- If there exists a Hadamard matrix of order m, then $(m^2, -1) \in S$.
- $(4m^4, 1) \in S$ for all positive integers m.
- $(4m^4, -1) \in S$ for all positive integers m.

4. Our results: (1) an new infinite family of thin divisible design graphs; (2) two new recursive constructions of RSHCDs; (3) orthogonal signings for infinitely many distance-regular graphs of diameter 3.

Our results (I)

First, let us give a construction for the matrices R based on RSHCDs.

Proposition 2

Let $t \geq 2$ be an integer. Let $u \geq 1$ be an integer such that there exists an RSHCDs of order $4u^2$ and type ε. Let T be the adjacency block matrix of a complete t-partite graph with parts of size $4u^2$ with zero blocks corresponding to the parts placed along the main diagonal of T. Let $H_1, H_2, \ldots, H_t$ be RSHCDs (possibly equal) of order $4u^2$, type ε and having -1's on the main diagonal. Let R be the matrix obtained from T by replacing the diagonal zero blocks with the $(0,2)$-matrices $H_1 + J$, $H_2 + J$, $\ldots$, $H_t + J$, where J is the all-ones matrix of order $4u^2$. Then R is a symmetric $(0,1,2)$-matrix of order $4tu^2$ without 2 on the main diagonal such that $R^2 = \alpha I + \beta J$, where $\alpha = 4u^2$ and $\beta = 4tu^2 - 4\varepsilon u$.

Our results (II)

Note that in the special case $t = 2$ in Proposition 2, we have a matrix $R = \begin{bmatrix} H_1 + J & J \\ J & H_2 + J \end{bmatrix}$ with $\alpha = 4u^2$ and $\beta = 8u^2 - 4\varepsilon u$.

In the next proposition we give another construction of a matrix R with the same parameters α and β.

Proposition 3

Let $u \geq 1$ be an integer. Let H be a regular Hadamard matrix of order $4u^2$ and such that $\frac{1}{2}(J + H)$ is the incidence matrix of a Menon design with parameters $(4u^2, 2u^2 + \delta u, u^2 + \delta u)$ where $\delta \in \{1, -1\}$ (or, equivalently, let H be a regular Hadamard matrix with row sum $2\delta u$). Then the matrix

$$R = \begin{bmatrix} J & J + H \\ (J + H)^T & J \end{bmatrix}$$

is a symmetric $(0, 1, 2)$-matrix of order $8u^2$, without 2 on the main diagonal such that $R^2 = \alpha I + \beta J$ where $\alpha = 4u^2$ and $\beta = 8u^2 + 4\delta u$.

Our results (III)

Further, let us describe some constructions for partner weighing matrices Q for the constructed matrices R.

Our results (IV)

Proposition 4

Let $u \geq 1$ be an integer. Let H be a symmetric Hadamard matrix of order y. Let C be a symmetric conference matrix of order $t \equiv 2 \pmod 4$ with all diagonal entries equal to 0. Then Kronecker product $C \otimes H$ is a symmetric weighing matrix of order yt and weight $y(t-1)$ having t zero blocks of size y on the main diagonal.

Note that a symmetric conference matrix of order $t \equiv 2 \pmod 4$ is known to exist if $t-1$ is a prime power. There are further constructions of symmetric conference matrices.

The resulting symmetric weighing matrices from Proposition 4 can be viewed as orthogonal signings of complete t-partite graphs with parts of size y. In particular, if $t = 2$, we get an orthogonal signing of a complete bipartite graph with parts of size y.

Our results (V)

The following proposition gives a characterisation of orthogonal signings of complete bipartite graphs.

Proposition 5

The existence of an orthogonal signing of a complete bipartite graph with parts of size y is equivalent to the existence of an Hadamard matrix of order y.

Proof.

The weighing matrix that is an orthogonal signing of a complete bipartite graph with parts of size y can always be written as $\begin{bmatrix} O & H \\ H^T & O \end{bmatrix}$, where H is an Hadamard matrix of order y. Conversely, if H is an Hadamard matrix of order y, then $\begin{bmatrix} O & H \\ H^T & O \end{bmatrix}$ is a symmetric weighing matrix that is an orthogonal signing of a complete bipartite graph with parts of size y. □

Our results (VI)

The following theorem gives infinitely many pairs of matrices (R, Q) that satisfy the conditions of Theorem 3.

Theorem 1

Let $t \geq 3$ be an integer. Let $u \geq 1$ be an integer. Let $H_1, H_2, \ldots, H_t$ be RSHCDs of order $4u^2$ and type ε. Let R be the corresponding $(0,1,2)$-matrix of order $4tu^2$ obtained according to Proposition 2. Let Q be a symmetric weighing matrix of order $4tu^2$ with t zero blocks of size $4u^2$ on the main diagonal, that is, an orthogonal signing of a complete t-partite graph with parts of size $4u^2$. Then the pair (R, Q) satisfies the conditions of Theorem 3 and thus produces a thin DDG with parameters $(8tu^2, 4tu^2 - 2\varepsilon u, 4u^2 - 2\varepsilon u, 2tu^2 - 2\varepsilon u, 4tu^2, 2)$.

Note that in the tuple of parameters in Theorem 1 we have $\lambda_1 = 4u^2 - 2\varepsilon u$, $\lambda_2 = 2tu^2 - 2\varepsilon u$, which implies $\lambda_1 \neq \lambda_2$ whenever $t \geq 3$, as was assumed. However, if $t = 2$, we would have equality $\lambda_1 = \lambda_2$.

Our results (VII)

Further we consider the case $t = 2$ that was excluded from Theorem 1 since, in view of Proposition 5, we can say more in this case and since we get two recursive constructions of RSHCDs deserving to be presented separately.

Our results (VIII)

Theorem 2

Let $u \geq 1$ be an integer. Let H_1, H_2 be RSHCDs of order $4u^2$ and type ε, having all diagonal entries equal to -1. Let $R = \begin{bmatrix} J+H_1 & J \\ J & J+H_2 \end{bmatrix}$ be the corresponding symmetric $(0,1,2)$-matrix of order $8u^2$ obtained according to Proposition 2 with $t = 2$. Let $Q = \begin{bmatrix} O & H \\ H^T & O \end{bmatrix}$ be a symmetric weighing matrix of order $8u^2$ and weight $4u^2$ with two zero blocks of size $4u^2$ on the main diagonal, that is, an orthogonal signing of a complete bipartite graph with parts of size $4u^2$, where H is an arbitrary Hadamard matrix. Then the pair (R, Q) satisfies the conditions of Theorem 3 and thus produces an almost proper thin DDG with parameters $(16u^2, 8u^2 - 2\varepsilon u, 4u^2 - 2\varepsilon u, 4u^2 - 2\varepsilon u, 8u^2, 2)$ and adjacency matrix $\frac{1}{2}\begin{bmatrix} J+H_1 & J-H & J+H_1 & J+H \\ (J-H)^T & J+H_2 & (J+H)^T & J+H_2 \\ J+H_1 & J+H & J+H_1 & J-H \\ (J+H)^T & J+H_2 & (J-H)^T & J+H_2 \end{bmatrix}$.

Our results (IX)

The almost proper thin DDGs from Theorem 2 are actually (v, k, λ)-graphs with parameters "RSHCD", that is, (v, k, λ)-graphs that can be converted into RSHCDs of order $16u^2$ and type ε. Thus, Theorem 2 can be interpreted as a recursive construction of RSHCDs where the input is two RSHCDs H_1, H_2 of order $4u^2$ and the same type ε and also an arbitrary Hadamard matrix H of order $4u^2$, and the output is the RSHCD

$$\begin{bmatrix} H_1 & -H & H_1 & H \\ -H^T & H_2 & H^T & H_2 \\ H_1 & H & H_1 & -H \\ H^T & H_2 & -H^T & H_2 \end{bmatrix}$$

of order $16u^2$ (that is, four times more) and the same type ε, or, equivalently,

$$\begin{bmatrix} H_1 & H_1 & -H & H \\ H_1 & H_1 & H & -H \\ -H^T & H^T & H_2 & H_2 \\ H^T & -H^T & H_2 & H_2 \end{bmatrix}.$$

Our results (X)

Moreover, if $H_1 = H_2 = H$, then the resulting adjacency matrix

$$\frac{1}{2}\begin{bmatrix} J+H & J-H & J+H & J+H \\ J-H & J+H & J+H & J+H \\ J+H & J+H & J+H & J-H \\ J+H & J+H & J-H & J+H \end{bmatrix}$$

corresponds to the RSHCD

$$\begin{bmatrix} H & -H & H & H \\ -H & H & H & H \\ H & H & H & -H \\ H & H & -H & H \end{bmatrix},$$

which is as the Kronecker product of the RSHCD

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{bmatrix}$$

of order 4 and positive type and the RSHCD H of type ε.

Our results (XI)

In a similar way, we get another recursive construction of RSHCDs, where the input is a regular Hadamard matrix H of order $4u^2$ and with row sum $2\delta u$ where $\delta \in \{1, -1\}$ and also two symmetric Hadamard matrices H_1, H_2 of order $4u^2$, with all diagonal elements equal to -1, and the output is the RSHCD

$$\begin{bmatrix} H_1 & H & -H_1 & H \\ H^T & H_2 & H^T & -H_2 \\ -H_1 & H & H_1 & H \\ H^T & -H_2 & H^T & H_2 \end{bmatrix}$$

of order $16u^2$ (that is, four times more) and type $-\delta$, or, equivalently,

$$\begin{bmatrix} H_1 & -H_1 & H & H \\ -H_1 & H_1 & H & H \\ H^T & H^T & H_2 & -H_2 \\ H^T & H^T & -H_2 & H_2 \end{bmatrix}.$$

Our results (XII)

Proposition 6 ([CH14, Proposition 4.1])

An (almost) proper thin DDG has a fixed-point free involution, and the orbits are the classes of the canonical partition.

In the special case that the DDG is almost proper, the DDG is a (v, k, λ)-graph. Then also the converse holds.

Proposition 7

If Γ *is a* (v, k, λ)*-graph admitting a fixed-point free involution, then with the partition into the orbits of the involution,* Γ *is an almost proper thin DDG.*

The two recursive constructions above thus produce (v, k, λ)-graphs admitting a fixed-point free involution interchanging only non-adjacent vertices.

Also, Proposition 7 provides another approach for constructing thin DDGs, by searching for fixed-point free involutions in (v, k, λ)-graphs.

Our results (XIII)

Theorem 3

If q is odd, the symplectic graph $Sp(4,q)$ has a fixed-point free involution, which interchanges only adjacent vertices.

Corollary 4

With the partition into the orbits of the involution from Theorem 3, the complement of $Sp(4,q)$ is an almost proper thin DDG with parameters $(v = q^3 + q^2 + q + 1, q^3, q^2(q-1), q^2(q-1), v/2, 2)$ such that the corresponding weighing matrix Q is the adjacency matrix of a signed distance-regular graph of diameter 3.

Thank you for your attention!