Erdős-Ko-Rado combinatorics of strongly regular graphs

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Strongly regular graphs

A graph Γ is called k-regular if there exists an integer $k \geq 0$ such that each vertex in Γ has exactly k neighbours.

A graph on v vertices is called a strongly regular graph with parameters (v, k, λ, μ) if:
(a) it is k-regular;
(b) each pair of adjacent vertices in the graph have exactly λ common neighbours;
(c) each pair of distinct nonadjacent vertices in the graph have exactly μ common neighbours.

Every non-trivial strongly regular graph has exactly three distinct eigenvalues k, r, s, where $s < 0 < r < k$ holds. Moreover, the eigenvalues r and s can be expressed in terms of the parameters (v, k, λ, μ).

Delsarte-Hoffman bound

Let s be the smallest eigenvalue of a k-regular strongly regular graph G. Delsarte proved [D73] that the clique number of G is at most

$$1 - \frac{k}{s}.$$

This bound is known as the Delsarte-Hoffman bound (see [BCN89, Proposition 1.3.2]).

A clique in a strongly regular graph whose size attains the Delsarte-Hoffman bound is called a Delsarte clique (see [H21] for historical remarks).

[BCN89] A. E. Brouwer, A. M. Cohen, and A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag, Berlin (1989).

[D73] P. Delsarte. *An algebraic approach to the association schemes of coding theory*, Philips Res. Rep. Suppl., (10):vi+97, 1973.

[H21] W. H. Haemers, *Hoffman's ratio bound*, Linear Algebra and its Applications Volume 617, (2021) 215–219. https://doi.org/10.1016/j.laa.2021.02.010

Erdős-Ko-Rado combinatorics of strongly regular graphs

The classical Erdős-Ko-Rado theorem deals with maximum cliques in the complement of the Kneser graphs and has two parts: the bound and the characterisation.

In Erdős-Ko-Rado combinatorics of strongly regular graphs, which was proposed by Chris Godsil and Karen Meagher, Kneser graphs are replaced with appropriate strongly regular graphs for whose vertices the notion of 'intersecting' can be defined, and the role of the required upper bound for the size of an intersecting family is played by the Delsarte-Hoffman bound.

Thus, the main goal in Erdős-Ko-Rado combinatorics of strongly regular graphs is to obtain a characterisation of Delsarte cliques in certain classes of strongly regular graphs. If such a characterisation is obtained, then it is interesting to determine the size of second largest maximal (w.r.t. inclusion) cliques and characterise them. Special attention is paid to block graphs of orthogonal arrays (including Paley graphs of square order) and block graphs of 2-designs.

Orthogonal arrays and their block graphs

An orthogonal array $OA(m,n)$ is an $m \times n^2$ array with entries from an n-element set T with the property that the columns of any $2 \times n^2$ subarray consist of all n^2 possible pairs.

The block graph of an orthogonal array $OA(m,n)$, denoted $X_{OA(m,n)}$, is defined to be the graph whose vertices are columns of the orthogonal array, where two columns are adjacent if there exists a row in which they have the same entry.

Let $S_{r,i}$ be the set of columns of $OA(m,n)$ that have the entry i in row r. These sets are cliques, and since each element of the n-element set T occurs exactly n times in each row, the size of $S_{r,i}$ is n for all i and r. These cliques are called the canonical cliques in the block graph $X_{OA(m,n)}$.

A simple combinatorial argument shows that the block graph of an orthogonal array is a strongly regular graph. Moreover, by the Delsarte-Hoffman bound, a clique in $X_{OA(m,n)}$ has size at most n, and the canonical cliques show the tightness of this bound.

EKR theorem for block graphs of orthogonal arrays: special case

The following theorem can be viewed as an analogue of the EKR theorem for block graphs of orthogonal arrays in a special case.

Theorem 1 ([GM15, Corollary 5.5.3])

Let $X = X_{OA(m,n)}$ be the block graph of an orthogonal array $OA(m,n)$ with $n > (m-1)^2$. Then the only maximum cliques in X are the canonical clique.

Comments:

- The theorem gives a characterisation of maximum cliques in terms of the parameters n and m of orthogonal arrays.
- There exist infinitely many orthogonal arrays for which this theorem does not apply to (for which the parameters n and m do not satisfy the condition $n > (m-1)^2$).

[GM15] C. D. Godsil, K. Meagher, *Erdős-Ko-Rado Theorems: Algebraic Approaches*, Cambridge University Press (2015).

EKR problems on block graphs of orthogonal arrays

Chris Godsil and Karen Meagher formulated the following open problems.

Problem 1 ([GM15, Problem 16.4.1])

Find a characterisation of the orthogonal arrays, based only on the parameters of the array, for which all of the maximum cliques in the orthogonal array graph are canonical cliques.

Problem 2 ([GM15, Problem 16.4.2])

Assume that $OA(m,(m-1)^2)$ *is an orthogonal array and its orthogonal array graph has non-canonical cliques of size* $(m-1)^2$*. Do these non-canonical cliques form subarrays?*

Problem 3 ([GM15, Section 16.4])

Determine all the maximum cliques in the block graph for any orthogonal array.

[GM15] C. D. Godsil, K. Meagher, *Erdős-Ko-Rado Theorems: Algebraic Approaches*, Cambridge University Press (2015).

Affine planes

An affine plane is a system of points and lines that satisfy the following axioms:

- Any two distinct points lie on a unique line.
- Given any line and any point not on that line there is a unique line which contains the point and does not meet the given line. (Playfair's axiom)
- There exist three non-collinear points (points not on a single line).

In an affine plane, two lines are called parallel if they are equal or disjoint. Using this definition, Playfair's axiom above can be replaced by:

- Given a point and a line, there is a unique line which contains the point and is parallel to the line.

Properties of finite affine planes and orthogonal arrays

If the number of points in an affine plane is finite, then if one line of the plane contains n points then:

- each line contains n points,
- each point is contained in $n+1$ lines,
- there are n^2 points in all, and
- there is a total of n^2+n lines.

The number n is called the order of the affine plane.

It is well-known that for an orthogonal array $OA(m,n)$, the inequality $m \leq n+1$ holds (note that an orthogonal array $OA(n+1,n)$ is called complete). Also, it is well-known that the existence of an affine plane of order n is equivalent to the existence of a complete orthogonal array $OA(n+1,n)$.

Since any non-empty subset of rows of an orthogonal array is again an orthogonal array, finite affine planes is a reach source of orthogonal arrays for all $m \in \{1,\ldots,n\}$. However, not every orthogonal array can be extended to a complete orthogonal array.

Desarguesian affine planes and orthogonal arrays

Let q be a prime power and W be a 2-dimensional vector space over the finite fields $\mathbb{F}_q$.

The space W can be viewed as a group w.r.t. to the addition of vectors. Then every 1-dimensional subspace forms a subgroup of order q in W.

Let P the set of vectors from W and L be the set of all cosets of 1-dimensional subspaces of W.

Then the pair (P, L), considered as the point set and the line set, with the natural incidence relation defined by containment, forms an affine plane of order q, denoted by $AG(2, q)$.

For every prime power q, the affine plane $AG(2, q)$ is called a Desarguesian affine plane. Moreover, any orthogonal array $OA(m, q)$ obtained as a subset of rows of a complete orthogonal array $OA(q+1, q)$ that is equivalent to a Desarguesian affine plane $AG(2, q)$, is also called Desarguesian.

Recent results on Desarguesian orthogonal arrays (I)

Towards Problem 1, we obtained [GY24] a characterisation of the Desarguesian orthogonal arrays, based only on the parameters of the array, for which all of the maximum cliques in the block graph of a Desarguesian orthogonal array are canonical cliques. We summarise the results in Theorem 2.

[GY24] S. Goryainov, C. H. Yip, *Extremal Peisert-type graphs without the strict-EKR property*, Journal of Combinatorial Theory, Series A, Volume 206, August 2024, 105887.

Recent results on Desarguesian orthogonal arrays (II)

Let $B(q)$ denote the smallest value of m such that there exists a Desarguesian orthogonal array $OA(m, q)$ whose block graph has a non-canonical maximum clique. If the block graph of a Desarguesian orthogonal array $OA(B(q), q)$ has a non-canonical maximum clique, we call such a graph extremal.

Theorem 2 ([GY24, Theorems 1.3–1.5, Example 5.6])

The following statements hold.
(1) *If q is prime and $q \geq 3$, then $B(q) = \frac{q+3}{2}$. Moreover, there exists a unique (up to isomorphism) extremal graph and its non-canonical cliques can be explicitly described.*
(2) *If $q = p^t$ for some prime p and integer $t > 1$, then $B(q) = p^{t-1} + 1$. Moreover, if q is a square or a cube, there exists a unique (up to isomorphism) extremal graph and its non-canonical cliques can be explicitly described.*
(3) *If $q = 32$, there exists at least two (up to isomorphism) extremal graphs.*

Recent results on Desarguesian orthogonal arrays (III)

Note that Theorem 2 contains some results towards Problem 2. Namely, in the case when q is a square, we proved the following about the extremal graph X_q:

- X_q is isomorphic to the affine polar graph $VO^+(4, \sqrt{q})$, an important graph in the theory of strongly regular graphs.
- Using the knowledge about affine polar graphs, we conclude:
    1. Every vertex of X_q is contained in $\sqrt{q}+1$ canonical maximum cliques and $\sqrt{q}+1$ non-canonical maximum cliques.
    2. Every non-canonical clique in X_q has the structure of a complete orthogonal array $OA(\sqrt{q}+1, \sqrt{q})$, which is widely known in finite geometry as a Baer subplane. In other words, the answer to the question in Problem 2 is positive at least in the case of Desarguesian orthogonal arrays.
    3. Canonical and non-canonical maximum cliques in X_q lie in the same orbit under the action of the automorphism group of X_q. This is quite surprising because one usually expects that canonical and non-canonical maximum intersecting families form separate classes and have essentially different structure.

Recent results on Desarguesian orthogonal arrays (IV)

Note that Theorem 2 also contains some results towards Problem 3. Namely, we determined all maximum cliques in block graphs of

- Desarguesian orthogonal arrays $OA(m, q)$, where $m < B(q)$, and
- Desarguesian orthogonal arrays $OA(B(q), q)$, where q is a prime, a square, or a cube.

Paley graphs of square order

Let u be a prime power, $u \equiv 1 \pmod 4$. Consider the finite field $\mathbb{F}_u$ and note that that exactly a half of elements from the multiplicative group $\mathbb{F}_u^*$ are squares, denote them by S_u. Define a graph $P(u)$ on the elements of $\mathbb{F}_u$ with two elements being adjacent whenever their difference belongs to S_u.

Paley graphs are known to be strongly regular.

If $u = q^2$ for some prime power q, we say that $P(q^2)$ is a Paley graph of square order. In this case, the Delsarte-Hoffman bound is equal to q and the subfield $\mathbb{F}_q$ is an example of a Delsarte clique.

The cliques that are affine images of $\mathbb{F}_q$ (that is, the cliques of the form $a\mathbb{F}_q + b$, where $a \in (\mathbb{F}_q^*)^2$ and $b \in \mathbb{F}_q$) are called the canonical cliques in $P(q^2)$.

In [B84], it was shown that Paley graphs of square order have only canonical maximum cliques.

[B84] A. Blokhuis, *On subsets of $GF(q^2)$ with square differences*, Indag. Math. 46 (1984) 369–372.

EKR problems on Paley graphs of square order

A vector in $\mathbb{R}^n$ is balanced if it is orthogonal to the all-ones vector $\mathbf{1}$. If v_S is the characteristic vector of a subset S of the set V, then we say that

$$v_S - \frac{|S|}{|V|}\mathbf{1}$$

is the balanced characteristic vector of S.

Chris Godsil and Karen Meagher asked for an alternative proof of the Blokhuis' result and formulated the following open problems.

Problem 4 ([GM15, Problem 16.5.1])

Show that the balanced characteristic vectors of the canonical cliques of $P(q^2)$, being $\frac{q-1}{2}$-eigenvectors of the adjacency matrix of $P(q^2)$, span the corresponding $\frac{q-1}{2}$-eigenspace.

Problem 5 ([GM15, Problem 16.5.2])

Prove that the only balanced characteristic vectors of sets of size q, in the $\frac{q-1}{2}$-eigenspace of $P(q^2)$, are the balanced characteristic vectors of the canonical cliques.

A solution for Problem 4

In [AGLY22], we pointed out that every Paley graph $P(q^2)$ of square order can be viewed as the block graph of an orthogonal array $OA(\frac{q+1}{2}, q)$. Moreover, we showed in [AGLY22] that the balanced characteristic vectors of canonical cliques in the block graph of any orthogonal array $OA(m, n)$ always span the $(n-m)$-eigenspace of the adjacency matrix, and thus solved Problem 4 in a broader context.

However, Problem 5 remains unsolved.

[AGLY22] S. Asgarli, S. Goryainov, H. Lin, C. H. Yip, *The EKR-module property of pseudo-Paley graphs of square order*, The Electronic Journal of Combinatorics, 29(4) (2022), #P4.33.

Comments on the Hilton-Milner theorem

- Similar to the EKR theorem, the Hilton-Milner theorem has two parts: a bound and a characterisation of families that meet the bound.
- In other words, the Hilton-Milner theorem shows what is second largest size of a maximal intersecting family (w.r.t. inclusion) and gives their characterisation.
- In terms of graphs, this theorem gives a characterisation of second largest maximal cliques in the in the complement of the Kneser graphs $K(n, k)$.

Hilton-Milner cliques in graphs

In terms of the complement of the Kneser graph $K(n,k)$, $2 \leq k \leq \frac{n}{2}$, the general second largest clique can be constructed as follows:

1. Take a canonical clique C.
2. Take a vertex x outside of C.
3. Let C_x be the set of neighbours of x in C.
4. The second largest clique is $\{x\} \cup C_x$ (and all second largest cliques have this structure).

Let X be a graph for which canonical cliques are defined. Then, for a canonical clique C and a vertex $x \notin C$, we call the clique $\{x\} \cup C_x$ a Hilton-Milner clique.

We are interested in the following natural problem.

Problem 6

Let $\{x\} \cup C_x$, denoted by D, be a Hilton-Milner clique in a graph X.
(1) *Is D maximal?*
(2) *If D is maximal, is D a second largest maximal clique?*
(3) *If D is not maximal, what are maximal extensions of D?*

Hilton-Milner cliques in the block graphs of Desarguesian orthogonal arrays

Theorem 3 (Brouwer, G., Shalaginov, Yip, 2024+)

Let $\{x\} \cup C_x$ be a Hilton-Milner clique in the block graph of a Desarguesian orthogonal array. Then the clique $\{x\} \cup C_x$ is not always maximal, but always has a unique maximal extension, where the size of this maximal extension may depend on the choice of C, but whenever C is fixed, the size does not depend on the choice of x.

Theorem 4 (Brouwer, G., Shalaginov, Yip, 2024+)

A second largest maximal clique in the block graph of a Desarguesian orthogonal array is not necessarily the maximal extension of an HM clique, but only finitely many counterexamples are known.

Conjecture 1 (Baker, Ebert, Hemmeter, Woldar, 1996)

The maximal extension of any Hilton-Milner clique in a Paley graph $P(q^2)$, where $q \geq 5$, is a second largest maximal clique of size $\frac{q+r(q)}{2}$, where $r(q)$ is the remainder of q modulo 4.

2-designs and their block graphs

A 2-$(n, m, 1)$ design is a collection of m-sets of an n-set with the property that every pair from the n-set is in exactly one set.

A specific 2-$(n, m, 1)$ design is denoted by $(V, \mathcal{B})$, where V is the n-set (which we call the base set) and $\mathcal{B}$ is the collection of m-sets — these are called the blocks of the design.

A 2-$(n, m, 1)$ design may also be called a 2-design. In a 2-design, any two blocks that intersect meet in exactly one point.

It is well-known that the number of blocks in a 2-$(n, m, 1)$ design is $\frac{n(n-1)}{m(m-1)}$ and each element of V occurs in exactly $\frac{n-1}{m-1}$ blocks.

The block graph of a 2-$(n, m, 1)$ design $(V, \mathcal{B})$ is the graph with the blocks of the design as the vertices in which two blocks are adjacent if and only if they intersect.

Cliques in the block graph $X_{(V,\mathcal{B})}$ are in one-to-one correspondence with intersecting set systems in $(V, \mathcal{B})$.

Canonical cliques in the block graphs of 2-designs

Fisher's inequality implies that the number of blocks in a 2-design is at least n; if equality holds, the design is said to be symmetric and the block graph of a symmetric 2-design is the complete graph K_n.

A simple combinatorial argument shows that the block graph of a 2-$(n, m, 1)$ design (that is not symmetric) is strongly regular. Moreover, by the Delsarte-Hoffman bound, a clique in the block graph of a 2-$(n, m, 1)$ design has size at most $\frac{n-1}{m-1}$, and the collection S_i of all blocks in the design that contain a given point i, is an example of a Delsarte clique. Such cliques S_i are called the canonical cliques of the block graph.

From this, we know that a set of intersecting blocks in a 2-design is no larger than the set of all blocks that contain a common point — this is the bound for an EKR-type theorem for the blocks in a design.

EKR theorem for block graphs of 2-designs: special case

It is not known in general for which designs the canonical intersecting sets are the only maximum intersecting sets. Chris Godsil and Karen Meagher offered a partial result.

Theorem 5 ([GM15, Corollary 5.3.5])

The only cliques of size $\frac{n-1}{m-1}$ *in the block graph* $X_{(V,\mathcal{B})}$ *of a* 2-$(n, m, 1)$ *design with* $n > m^3 - 2m^2 + 2m$ *are the canonical cliques.*

The characterisation in Theorem 5 may fail if $n \leq m^3 - 2m^2 + 2m$.

Theorem 6 ([GM15, Exercise 5.7])

In case $n = m^3 - 2m^2 + 2m$*, a non-canonical clique in the block graph of a* 2-$(n, m, 1)$ *design necessarily forms a* $(m^2 - m + 1, m, 1)$ *subdesign (which is a projective plane of order* $m - 1$*).*

EKR problems on 2-designs

Problem 7 ([GM15, Problem 16.3.1])

Determine a characterisation of the 2-$(n, m, 1)$ *designs, based only on the parameters of the design, for which the only maximum cliques in the block graph are the canonical cliques.*

The line graphs of the projective spaces $PG(3, q)$ give infinitely many examples of designs whose block graphs have non-canonical maximum cliques with the structure of a subdesign. It was not clear if this a result of a wider phenomenon.

Problem 8 ([GM15, Problem 16.3.2])

When the block graph of a design has maximum cliques that are not canonical, are the non-canonical cliques isomorphic to smaller designs?

Problem 9 ([GM15, Section 16.3])

Determine all the maximum cliques in the block graph for any t-(n, m, λ) *design.*

A solution for Problem 8

In [GK23], we negatively answered the question from Problem 8 by showing that the 2-$(66, 6, 1)$ design constructed in [D80] has non-canonical maximum cliques without a subdesign structure and is a smallest (w.r.t. the number of points) known such a design. However, only finitely many examples are known. If one wants an infinite family of such examples, the block size cannot be a constant in this family. Only four infinite families of 2-designs with a growing block size are known and only for one of them non-canonical maximum cliques without a subdesign structure might exist, namely, this is the family of Denniston 2-$(2^{a+b} + 2^a - 2^b, 2^a, 2)$ designs, existing for all integral a, b, where $2 \leq a < b$. Computations show that only canonical maximum cliques exist in the block graphs of such designs for small admissible values of a, b.

[D80] R. H. F. Denniston, *A Steiner system with a maximal arc*, Ars Combin. 9 (1980) 247–248.

[GK23] S. Goryainov, E. V. Konstantinova, *Non-canonical maximum cliques without a design structure in the block graphs of 2-designs*, arXiv:2311.01190.

Possible directions for further research

1. For a better understanding of small orthogonal arrays, it is possible to examine the database of small non-Desarguesian projective planes by Eric Moorhouse (the deletion of a line from a projective plane together with its points results in an affine plane, which depends on the choice of the deleted line; further, these affine planes, considered as complete orthogonal arrays, give rise to a plenty of orthogonal arrays with various parameters). Some of these non-Desarguesian projective planes are parts of infinite families.
2. For 2-designs, it is possible to examine all known constructions and try to produce more examples of 2-designs whose block graphs have non-canonical maximum cliques without a subdesign structure. In particular, it would be interesting to obtain an answer for the Denniston designs. In general, the following question is interesting: does there exist infinitely many 2-designs whose block graphs have non-canonical maximum cliques without a subdesign structure?

Thank you for your attention!