On the weight-distribution bound for eigenfunctions of strongly regular graphs

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Outline

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2. Tightness of the weight-distribution bound for some collinearity graphs of Desarguesian nets:
   - Paley graphs of square order;
   - generalised Paley graphs of square order;
   - hyperbolic affine polar graphs in dimension 4.
3. An geometric problem related to Desarguesian nets.
4. Tightness of the weight-distribution bound for the line graphs of projective and affine spaces.
5. Two applications in algebraic graph theory:
   - Haemers' construction of strongly regular graphs based on generalised quadrangles associated with a hyperoval at infinity;
   - special spreads in 4-dimensional symplectic spaces and a related construction of divisible design graphs

1. Introduction to the weight-distribution bound

Eigenfunctions of graphs

Let $\Gamma = (V, E)$ be a k-regular graph on n vertices and λ be an eigenvalue of its adjacency matrix A. Let $u = (u_1, \ldots, u_n)^t$ be an eigenvector of A corresponding to λ. Then u defines a function $f_u : V \mapsto \mathbb{R}$, which is called a λ-eigenfunction of Γ.

For an eigenfunction f_u of Γ, the *support* is the set

$$Supp(f_u) := \{x \in V \mid f_u(x) \neq 0\}.$$

MS-problem

The following problem was first formulated in [VK15] (see also a survey [SV21] for the motivation and details).

Problem 1 (MS-problem)

Given a graph Γ *and its eigenvalue* λ*, find the minimum cardinality of support of a* λ*-eigenfunction of* Γ*.*

A λ-eigenfunction having the minimum cardinality of support is called optimal.

Problem 2 (Strong MS-problem)

Given a graph Γ *and its eigenvalue* λ*, characterise optimal* λ*-eigenfunctions of* Γ*.*

[VK15] K. V. Vorobev, D. S. Krotov, *Bounds for the size of a minimal 1-perfect bitrade in a Hamming graph*, Journal of Applied and Industrial Mathematics 9(1) (2015) 141–146.

[SV21] E. Sotnikova, A. Valyuzhenich, *Minimum supports of eigenfunctions of graphs: a survey*, Art Discrete Appl. Math. 4 (2021), no. 2, Paper No. 2.09, 34 pp.

A survey on the strong MS-problem

Recently, Problem 2 was solved for several classes of graphs:

- all eigenvalues of Hamming graphs $H(n, q)$ when $q = 2$ or $q > 4$ and some eigenvalues of $H(n, q)$ when $q = 3, 4$;
- all eigenvalues of Johnson graphs (asymptotically);
- the smallest eigenvalue of Hamming, Johnson and Grassmann graphs;
- the smallest eigenvalue of the collinearity graphs of affine spaces $\mathrm{AG}(n, q)$;
- the largest non-principal eigenvalue of Star graphs S_n for all $n \geq 8$;
- the largest non-principal eigenvalue of Doob graphs;
- the positive non-principal eigenvalue of the strongly regular polar graphs, including affine polar graphs;
- the negative eigenvalue of the symplectic strongly regular graphs in dimension 4.

A survey on MS-problem

Except for the results from the previous slide, Problem 1 was solved for several more classes of graphs:

- the negative eigenvalue of (2×2) bilinear forms graphs over a prime field;
- both non-principal eigenvalues of Paley graphs of square order;
- the negative eigenvalue of generalised Paley graphs that are collinearity graphs of Desarguesian nets;
- the negative eigenvalue of the unitary polar graphs in dimension 4;
- the negative eigenvalue of the affine polar graphs in dimension 4.

Weight-distribution bound

Let Γ be a distance-regular graph of diameter $D(\Gamma)$ with intersection array $(b_0, b_1, \ldots, b_{D(\Gamma)-1}; c_1, c_2, \ldots, c_{D(\Gamma)})$ and nonprincipal eigenvalue λ.

Theorem 1 (Weight-distribution bound, [KMP16, Corollary 1])

A λ-eigenfunction f of Γ has at least $\sum_{i=0}^{D(G)} |W_i|$ nonzeros, where

$$W_0 = 1,$$

$$W_1 = \lambda$$

and

$$W_i = \frac{(\lambda - a_{i-1})W_{i-1} - b_{i-2}W_{i-2}}{c_i}.$$

[KMP16] D. S. Krotov, I. Yu. Mogilnykh, V. N. Potapov, *To the theory of q-ary Steiner and other-type trades*, Discrete Mathematics 339 (3) (2016) 1150–1157.

Known results when the weight-distribution bound is tight

- the eigenvalue -1 of the Boolean Hamming graph of an odd dimension and the minimum eigenvalue of an arbitrary Hamming graph;
- both non-principal eigenvalues of Paley graphs of square order;
- the negative eigenvalue of generalised Paley graphs that are the collinearity graphs of Desarguesian nets
- the minimum eigenvalue of Johnson graphs;
- the minimum eigenvalue of Grassmann graphs;
- the minimum eigenvalue of strongly regular bilinear forms graphs over a prime field.

Tightness of WDB for the smallest eigenvalue of a DRG

It was shown in [KMP16] that, for the smallest eigenvalue of a distance-regular graph Γ, the tightness of the weight-distribution bound implies the existence of an isometric bipartite distance-regular induced subgraph $T_0 \cup T_1$, where T_0 and T_1 are parts, such that an optimal eigenfunction, up to multiplication by a non-zero constant, has the following form:

$$f(x) = \begin{cases} 1, & \text{if } x \in T_0; \\ -1, & \text{if } x \in T_1; \\ 0, & \text{otherwise.} \end{cases}$$

The converse is known to be true if Γ has Delsarte cliques and every edge of Γ lies in a constant number of Delsarte cliques.

Conjecture 1 (Krotov)

The converse is always true.

[KMP16] D. S. Krotov, I. Yu. Mogilnykh, V. N. Potapov, *To the theory of q-ary Steiner and other-type trades*, Discrete Mathematics 339 (3) (2016) 1150–1157.

Tightness of the weight-distribution bound for a non-principal eigenvalue of an SRG

If Γ is a strongly regular graph with non-principal eigenvalues θ, τ, where $\tau < 0 < \theta$, the following holds.

Theorem 2 ([KMP16], Weight-distribution bound for SRG)

(1) *An* τ*-eigenfunction* f *of* Γ *has at least* (-2τ) *nonzeros. If* $|Supp(f)|$ *meets the bound, then there exists an induced complete bipartite subgraph with parts* T_0, T_1 *of size* $-\tau$.
(2) *An* θ*-eigenfunction* f *of* Γ *has at least* $2(\theta+1)$ *nonzeros; if* $|Supp(f)|$ *meets the bound, then there exists an induced disjoint union of two cliques* T_0, T_1 *of size* $\theta+1$.

[KMP16] D. S. Krotov, I. Yu. Mogilnykh, V. N. Potapov, *To the theory of q-ary Steiner and other-type trades*, Discrete Mathematics 339 (3) (2016) 1150–1157.

2. Tightness of the weight-distribution bound for some collinearity graphs of Desarguesian nets

Tightness of the weight-distribution bound for Paley graphs of square order (I)

In [GKSV18], for Paley graphs $P(q^2)$, we showed the tightness of the weight-distribution bound for both non-principal eigenvalues, which are $\tau = \frac{-1-q}{2}$ and $\theta = \frac{-1+q}{2}$.

Let β be a primitive element in $\mathbb{F}_{q^2}$. Put $\omega := \beta^{q-1}$. Then $Q = \langle \omega \rangle$ is the subgroup of order $q+1$ in $\mathbb{F}_{q^2}^*$.

Facts about Q:

- Q is an oval in the corresponding affine plane;
- Q is the kernel of the norm mapping $N : \mathbb{F}_{q^2}^* \mapsto \mathbb{F}_q^*$, which means that $Q = \{\gamma \in \mathbb{F}_{q^2}^* \mid \gamma^{q+1} = 1\}$, or, equivalently, $Q = \{x + y\alpha \mid x, y \in \mathbb{F}_q, x^2 - y^2 d = 1\}$, where d is a non-square in $\mathbb{F}_q^*$ and $\alpha^2 = d$.

[GKSV18] S. Goryainov, V. Kabanov, L. Shalaginov, A. Valyuzhenich, *On eigenfunctions and maximal cliques of Paley graphs of square order*, Finite Fields and Their Applications 52 (2018) 361–369.

Tightness of the weight-distribution bound for Paley graphs of square order (II)

Let $Q_0 = \langle \omega^2 \rangle$ and $Q_1 = \omega Q_0$.

Facts about Q:

- if $q \equiv 1(4)$, then $Q = Q_0 \cup Q_1$ induces a complete bipartite graph with parts Q_0 and Q_1;
- if $q \equiv 3(4)$, then $Q = Q_0 \cup Q_1$ induces a pair of disjoint cliques Q_0 and Q_1.

Corollary 1 ([GKSV18])

The weight-distribution bound is tight for both non-principal eigenvalues of Paley graphs of square order.

Knowing the structure of Q, we were also able to construct new maximal cliques of the second largest known size in Paley graphs of square order (see [GKSV18]).

[GKSV18] S. Goryainov, V. Kabanov, L. Shalaginov, A. Valyuzhenich, *On eigenfunctions and maximal cliques of Paley graphs of square order*, Finite Fields and Their Applications 52 (2018) 361–369.

Generalised Paley graphs of square order; WDB for the smallest eigenvalue

Let $m > 1$ be a positive integer. Let q be an odd prime power, $q \equiv 1\ (2m)$. The m-Paley graph on $\mathbb{F}_q$, denoted $GP(q, m)$, is the Cayley graph $Cay(\mathbb{F}_q^+, (\mathbb{F}_q^*)^m)$, where $(\mathbb{F}_q^*)^m$ is the set of m-th powers in $\mathbb{F}_q^*$.

We consider the graphs $GP(q^2, m)$, where q is an odd prime power and m divides $q + 1$; these graphs are strongly regular and form a generalisation of Paley graphs of square order (the usual Paley graphs of square order are just 2-Paley graphs of square order).

The eigenvalues of $GP(q^2, m)$ are $\tau = (-\frac{q+1}{m})$ and $\theta = \frac{(m-1)q-1}{m}$.

Given an odd prime power q and an integer $m > 1$ such that m divides $q + 1$, a $(-\frac{q+1}{m})$-eigenfunction of the generalised Paley graph $GP(q^2, m)$ has at least $\frac{2(q+1)}{m}$ non-zeroes.

Structure of Q (I)

Let us divide Q into m parts

$$Q = Q_0 \cup Q_1 \cup \ldots \cup Q_{m-1},$$

where $Q_0 = \langle \omega^m \rangle$, $Q_1 = \omega Q_0$, $\ldots$, $Q_{m-1} = \omega^{m-1} Q_0$.

Proposition 1 ([GSY23])

Let q be a prime power and m be an integer such that $m > 1$, m divides $q+1$. The mapping $\gamma \mapsto \gamma^{q-1}$ is a homomorphism from $\mathbb{F}_{q^2}^*$ to Q. Moreover, an element γ is an m-th power in $\mathbb{F}_{q^2}^*$ if and only if γ^{q-1} is an m-th power in Q.

[GSY23] S. Goryainov, L. Shalaginov, C. H. Yip, *On eigenfunctions and maximal cliques of generalised Paley graphs of square order*, Finite Fields and Their Applications, Volume 87, March 2023, 102150.

Structure of Q (II)

Let us divide Q into m parts

$$Q = Q_0 \cup Q_1 \cup \ldots \cup Q_{m-1},$$

where $Q_0 = \langle \omega^m \rangle$, $Q_1 = \omega Q_0$, …, $Q_{m-1} = \omega^{m-1} Q_0$.

Proposition 2 ([GSY23])

Let γ be an arbitrary element from Q, $\gamma \neq 1$. Then, for the image of $(\gamma - 1)$ under the action of the homomorphism, the following equality holds:

$$(\gamma - 1)^{q-1} = -\frac{1}{\gamma}.$$

[GSY23] S. Goryainov, L. Shalaginov, C. H. Yip, *On eigenfunctions and maximal cliques of generalised Paley graphs of square order*, Finite Fields and Their Applications, Volume 87, March 2023, 102150.

Structure of Q (III)

The following theorem basically states that each of the sets $Q_0, Q_1 \ldots, Q_{m-1}$ induces either a clique or an independent set, and there are at most two cliques among them.

Moreover, the theorem states that for every independent set Q_{i_1}, there exists uniquely determined independent set Q_{i_2} among $Q_0, Q_1 \ldots, Q_{m-1}$ such that there are all possible edges between Q_{i_1} and Q_{i_2} and there are no edges between Q_{i_1} and $Q \setminus Q_{i_2}$.

Structure of Q (IV)

Theorem 4.6 ([GSY22, Theorem 4.1])

Given an odd prime power q and an integer $m > 1$, m divides $q + 1$, the following statements hold for the subgraph of $GP(q^2, m)$ induced by Q.

(1) If m divides $\frac{q+1}{2}$ and m is odd, then Q_0 is a clique, and $Q_1, \ldots, Q_{m-1}$ are independent sets; moreover, for any distinct i_1, i_2 such that $0 \leq i_1 < i_2 \leq m - 1$, there are all possible edges between the sets Q_{i_1} and Q_{i_2} if $i_1 + i_2 \equiv 0 \pmod{m}$, and there are no such edges if $i_1 + i_2 \not\equiv 0 \pmod{m}$. In particular, $Q_1 \cup Q_{m-1}, \ldots, Q_{\frac{m-1}{2}} \cup Q_{\frac{m+1}{2}}$ induce $\frac{m-1}{2}$ complete bipartite graphs.

[GSY23] S. Goryainov, L. Shalaginov, C. H. Yip, *On eigenfunctions and maximal cliques of generalised Paley graphs of square order*, Finite Fields and Their Applications, Volume 87, March 2023, 102150.

Structure of Q (V)

(2) If m divides $\frac{q+1}{2}$ and m is even, then $Q_0, Q_{\frac{m}{2}}$ are cliques, and $Q_1, \ldots, Q_{\frac{m}{2}-1}, Q_{\frac{m}{2}+1}, \ldots, Q_{m-1}$ are independent sets; moreover, for any distinct i_1, i_2 such that $0 \leq i_1 < i_2 \leq m-1$, there are all possible edges between the sets Q_{i_1} and Q_{i_2} if

$$i_1 + i_2 \equiv 0 \pmod{m} \text{ and } \{i_1, i_2\} \neq \{0, \frac{m}{2}\}$$

and there are no such edges if

$$i_1 + i_2 \not\equiv 0 \pmod{m} \text{ or } \{i_1, i_2\} = \{0, \frac{m}{2}\}.$$

In particular, $Q_1 \cup Q_{m-1}, \ldots, Q_{\frac{m}{2}-1} \cup Q_{\frac{m}{2}+1}$ induce $(\frac{m}{2} - 1)$ complete bipartite graphs.

Structure of Q (VI)

(3) If m does not divide $\frac{q+1}{2}$, then m is even.
(3.1) If $\frac{m}{2}$ is odd, then $Q_0, Q_1, \ldots, Q_{m-1}$ are independent sets; moreover, for any distinct i_1, i_2 such that $0 \leq i_1 < i_2 \leq m-1$, there are all possible edges between the sets Q_{i_1} and Q_{i_2} if

$$i_1 + i_2 \equiv \frac{m}{2} \pmod{m},$$

and there are no such edges if

$$i_1 + i_2 \not\equiv \frac{m}{2} \pmod{m}.$$

In particular, if $m = 2$, $Q = Q_0 \cup Q_1$ is a complete bipartite graph; if $m \geq 6$,
$Q_0 \cup Q_{\frac{m}{2}}, \ldots, Q_{\frac{m-2}{4}} \cup Q_{\frac{m+2}{4}}, Q_{\frac{m}{2}+1} \cup Q_{m-1}, \ldots, Q_{\frac{3m-2}{4}} \cup Q_{\frac{3m+2}{4}}$
induce $\frac{m}{2}$ complete bipartite graphs.

Structure of Q (VII)

(3.2) If $\frac{m}{2}$ is even, then $Q_{\frac{m}{4}}, Q_{\frac{3m}{4}}$ are cliques, and $Q_0, \ldots, Q_{\frac{m}{4}-1}, Q_{\frac{m}{4}+1}, \ldots, Q_{\frac{3m}{4}-1}, Q_{\frac{3m}{4}+1}, \ldots, Q_{m-1}$ are independent sets; moreover, for any distinct i_1, i_2 such that $0 \leq i_1 < i_2 \leq m-1$, there are all possible edges between the sets Q_{i_1} and Q_{i_2} if

$$i_1 + i_2 \equiv \frac{m}{2} \pmod{m} \text{ and } \{i_1, i_2\} \neq \{\frac{m}{2}, \frac{3m}{2}\},$$

and there are no such edges if

$$i_1 + i_2 \not\equiv \frac{m}{2} \pmod{m} \text{ or } \{i_1, i_2\} = \{\frac{m}{2}, \frac{3m}{2}\}.$$

In particular, if $m = 4$, $Q_0 \cup Q_2$ is a complete bipartite graph; if $m \geq 8$, then
$Q_0 \cup Q_{\frac{m}{2}}, \ldots, Q_{\frac{m-4}{4}} \cup Q_{\frac{m+4}{4}}, Q_{\frac{m}{2}+1} \cup Q_{m-1}, \ldots, Q_{\frac{3m-4}{4}} \cup Q_{\frac{3m+4}{4}}$
induce $\frac{m-2}{2}$ complete bipartite graphs.

Structure of Q (VIII) and tightness of WDB for the smallest eigenvalue of $GP(q^2, m)$

Corollary 4.7 ([GSY23)

Let q be an odd prime power and m be an integer $m \geq 2$, m divides $q+1$. Then, except for the case $m = 2$ and 2 divides $\frac{q+1}{2}$, there is at least one pair Q_{i_1}, Q_{i_2} among $Q_0, \ldots, Q_{m-1}$ such that $Q_{i_1} \cup Q_{i_2}$ induces a complete bipartite subgraph.

Corollary 4.8 ([GSY23)

Let q be an odd prime power and m be an integer $m \geq 2$, m divides $q+1$. Then the weight-distribution bound is tight for the eigenvalue $(-\frac{q+1}{m})$ of $GP(q^2, m)$.

[GSY23] S. Goryainov, L. Shalaginov, C. H. Yip, *On eigenfunctions and maximal cliques of generalised Paley graphs of square order*, Finite Fields and Their Applications, Volume 87, March 2023, 102150.

4-dimensional hyperbolic affine polar graphs as collinearity graphs of Desarguesian nets (I)

Let $q = r^2$ and $Q = \{\gamma \in \mathbb{F}_q^* \mid \gamma^{r+1} = 1\}$.

Assume $\mathbb{F}_{q^2} = \{x + y\beta : x, y \in \mathbb{F}_q\}$, where β is a root of an irreducible polynomial $f(t) = t^2 + dt + e \in \mathbb{F}_q[t]$.

Identify the points of $AG(2, q)$ with elements of $\mathbb{F}_{q^2}$ as $(x, y) \longleftrightarrow x + y\beta$.

Let N_r be the net of degree $r + 1$ generated by the set of lines $\bigcup_{\delta \in Q} \{(\delta + \beta)\mathbb{F}_q^*\}$.

4-dimensional hyperbolic affine polar graphs as collinearity graphs of Desarguesian nets (II)

Theorem 3 ([GY24])

The following statements hold.
(1) *The collinearity graph of* N_r *is isomorphic to the hyperbolic affine polar graph* $VO^+(4, r)$.
(2) *If* $T_0 = Q$ *and* $T_1 = Q\beta$, *then the function* $f : \mathbb{F}_{q^2} \mapsto \mathbb{R}$ *defined by the rule*

$$f(\gamma) = \begin{cases} 1, & \gamma \in T_0; \\ -1, & \gamma \in T_1; \\ 0, & \gamma \notin T_0 \cup T_1, \end{cases}$$

is an eigenfunction of $VO^+(4, r)$ *corresponding the negative eigenvalue* $-r - 1$ *and achieving the weight-distribution bound.*

Note that T_0 and T_1 are subsets of size $r + 1$ of two lines in $AG(2, q)$ that do not belong to the net N_r, that is, are cocliques. Also, the subgraph induced by $T_0 \cup T_1$ is a complete bipartite subgraph with parts T_0 and T_1.

3. An geometric problem related to Desarguesian nets

(m, m)-configurations in affine planes

Let q be a prime power. Let T_0, T_1 be two disjoint subsets of points of the affine plane $\mathrm{AG}(2, q)$ such that $|T_0| = |T_1| = m$ for some integer m. The pair (T_0, T_1) is said to be an (m, m)-configuration if there exist a subnet N in $\mathrm{AG}(2, q)$ of degree m such that every line connecting a point from T_0 and a point from T_1 does not contain other points from $T_0 \cup T_1$ and belongs to the net N.

It follows that, for an (m, m)-configuration, the net N is uniquely determined, an we then say that the net N is induced by this (m, m)-configuration. Moreover, for every point p of T_0 (resp. T_1), each line from the pencil of N-lines passing through p contains exactly one point from T_1 (resp. T_0).

Note that for a net of degree m, the smallest eigenvalue of its collinearity graph is $-m$. It is not difficult to see that WDB is tight for the eigenvalue $-m$ of the collinearity graph of a net of degree m if and only if this net is induced by an (m, m)-configuration. In other words, Krotov's conjecture is true for the collinearity graphs of nets.

Problem

Paley graphs of square order, generalised Paley graphs of square order and affine polar graphs $VO^+(4, r)$ provide us with nice non-trivial examples, and it is natural to ask for more.

Problem 3

What are (m, m)-configurations in finite affine planes? In particular, what are (m, m)-configurations in the Desarguesian planes?

4. Tightness of the weight-distribution bound for the line graphs of projective and affine spaces

Grassmann graphs and dual polar graphs (I)

Let $\mathbb{F}_q^n$ be an n-dimensional vector space over the field $\mathbb{F}_q$ of order q. The Grassmann graph $J_q(n, d)$ is defined as follows. The vertices are the d-dimensional subspaces of $\mathbb{F}_q^n$. Two vertices are adjacent whenever they intersect in a $(d-1)$-dimensional subspace. The Grassmann graph is a distance-transitive graph.

Before formulating a theorem, we briefly introduce the dual polar graph $D_d(q)$, which plays the role of the induced bipartite subgraph for $J_q(n, d)$.

Let Q be a nondegenerate hyperbolic quadratic form. The dual polar graph $D_d(q)$ has as vertices the d-dimensional totally isotropic subspaces, with respect to Q; two vertices α and β are adjacent whenever $\dim(\alpha \cap \beta) = d - 1$.

Grassmann graphs and dual polar graphs (II)

Theorem 4 ([KMP16])

The minimum cardinality of support of an eigenfunction corresponding to the minimum eigenvalue of $J_q(n,d)$ *is* $\prod_{i=1}^{d}(q^{d-i}+1)$, *which is also equal to WDB, and the corresponding bipartite distance-regular subgraph of* $J_q(n,d)$ *has the parameters of the dual polar graph* $D_d(q)$.

Theorem 5 ([C98])

For the minimum eigenvalue of $J_q(n,d)$ *every optimal eigenfunction is induced by an embedding of* $D_d(q)$ *as an induced subgraph.*

[KMP16] D. S. Krotov, I. Yu. Mogilnykh, V. N. Potapov, *To the theory of q-ary Steiner and other-type trades*, Discrete Mathematics 339 (3) (2016) 1150–1157.

[C98] S. Cho. *Minimal null designs of subspace lattice over finite fields*, Linear Algebra Appl., 282 (1–3):199–220, 1998.

Grassmann graphs and dual polar graphs (III)

If $d = 2$, that is $J_q(n, d)$ is a strongly regular graph defined on the projective lines, then the embedded induced dual polar graphs just correspond to reguli.

A Cameron-Liebler line class in $\mathrm{PG}(3, q)$ is a part of an equitable 2-partition of the graph defined on the lines (adjacent when intersecting), corresponding to the positive non-principal eigenvalue.

Since every equitable 2-partition of a regular graph is equivalent to an eigenfunction taking exactly two values and since eigenfunctions corresponding to different eigenvalues are orthogonal, it might be possible to use $(1, -1, 0)$-eigenfunctions to restrict the behavior of Cameron-Liebler line classes. However, for the reguli eigenfunctions this restriction only tells that, for each regulus, every Cameron-Liebler line class shares the same number of lines with this regulus and its opposite. This condition is famously known to be one of the equivalent definitions of Cameron-Liebler line classes.

[GP24] S. Goryainov, D. Panasenko, *On eigenfunctions of the block graphs of geometric Steiner systems*, Journal of Combinatorial Designs, Volume 32, Issue 11,

Affine spaces

Let $X_q(n, 1)$ denote the graph defined on lines of $AG(n, q)$, $n \geq 3$, adjacent when intersecting.

Theorem 6 ([GP24])

The weight-distribution bound is tight for the eigenvalue $-q$ of $X_q(n, 1)$. Moreover, there are exactly two types of optimal $-q$-eigenfunctions: there is a one-to-one correspondence between optimal eigenfunctions of the first type and pairs of parallel classes of lines from some subplane $AG(2, q)$; there is a one-to-one correspondence between optimal eigenfunctions of the second type and affine reguli in three-dimensional subspaces of $AG(n, q)$.

[GP24] S. Goryainov, D. Panasenko, *On eigenfunctions of the block graphs of geometric Steiner systems*, Journal of Combinatorial Designs, Volume 32, Issue 11, November 2024, Pages 629-641.

Possible application to Cameron-Liebler line classes in $\mathrm{AG}(3, q)$

In [DMSS20], Cameron–Liebler line classes in $\mathrm{AG}(3, q)$ were introduced.

Problem 4

Do the optimal eigenfunctions of $X_q(n, 1)$ *lead to an equivalent definition of Cameron-Liebler line classes in* $\mathrm{AG}(3, q)$ *just the optimal (reguli) eigefunctions of Grassmann graphs do?*

[DMSS20] J. D'haeseleer, J. Mannaert, L. Storme, and A. Švob, *Cameron–Liebler line classes in* $\mathrm{AG}(3, q)$, Finite Fields Appl. 67 (2020), 101706.

5. Two applications in algebraic graph theory

Two applications in algebraic graph theory

In his PhD thesis, Willem Haemers gave a new construction of strongly regular graphs, which was based on the line graph of the generalised quadrangle associated with a hyperoval at infinity. The key step of the construction was the partition of the graph into induced complete bipartite subgraphs (which in turn correspond to optimal eigenfunctions) and reversing the adjacency within each of these subgraphs.

In [DGHS25], we used a similar idea to produce new constructions of divisible design graphs.

[DGHS25] B. De Bruyn, S. Goryainov, W. H. Haemers, L. Shalaginov, *Divisible design graphs from the symplectic graph*, Designs, Codes and Cryptography, Volume 93, 1401–1424 (2025).

Thank you for your attention!