Several results on cliques in strongly regular graphs

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The International Conference and PhD-Master Summer School on Graphs and Groups, Representations and Relations (G2R2)
August 12th, 2018

Outline

- Result 1 on maximal cliques in Paley graphs of square order
- Result 2 on equitable partitions of Latin-square graphs
- Result 3 on Neumaier graphs

Result 1. Outline

1. Preliminary
2. Two families of maximal cliques in Paley graphs of square order
3. Eigenfunctions of Paley graphs of square order having minimum cardinality of support

Paley graph $P(q_1)$

Let q_1 be an odd prime power, $q_1 \equiv 1(4)$.

The Paley graph of order q_1 (denoted by $P(q_1)$) is a graph defined as follows:

- the vertex set is the finite field $\mathbb{F}_{q_1}$;
- two vertices γ_1, γ_2 are adjacent iff $\gamma_1 - \gamma_2$ is a square in $\mathbb{F}^*_{q_1}$.

Since -1 is a square in $\mathbb{F}^*_{q_1}$ iff $q_1 \equiv 1(4)$, the graph $P(q_1)$ is undirected.

Maximum and maximal cliques in $P(q_1)$

A clique (coclique) in an undirected graph is a set of pairwise adjacent (non-adjacent) vertices.

Problem 1.1

What are maximum and what are maximal but not maximum cliques (cocliques) in $P(q_1)$?

Since $P(q_1)$ is self-complementary, the studying cliques and the studying cocliques in $P(q_1)$ are equivalent.

Since $P(q_1)$ is strongly regular, we can apply Delsarte-Hoffman bound to $P(q_1)$. It says that a clique (coclique) in $P(q_1)$ has at most $\sqrt{q_1}$ vertices.

Maximal cliques in $P(q_1)$

Problem 1.1 is unsolved in general.

In 2016, Greaves and Soicher improved [1] the Delsarte-Hoffman bound for infinitely many feasible parameter tuples for strongly regular graphs, including infinitely many parameter tuples that correspond to Paley graphs.

[1] G. R. W. Greaves, L. H. Soicher, *On the clique number of a strongly regular graph*, arXiv:1604.08299, April 2016.

The case of Paley graphs of square order $q_1 = q^2$

Suppose $q_1 = q^2$.

According to the Delsarte-Hoffman bound, a clique in $P(q^2)$ has at most q vertices.

Since every element from $\mathbb{F}_q^*$ is a square in $\mathbb{F}_{q^2}^*$, the subfield $\mathbb{F}_q$ induces a clique of size q in $P(q^2)$, which implies the tightness of the Delsarte-Hoffman bound.

In 1984, Blokhuis classified maximum cliques in $P(q^2)$ and proved [2] that such a clique is an affine image of the subfield $\mathbb{F}_q$.

[2] A. Blokhuis, *On subsets of $GF(q^2)$ with square differences*, Indag. Math. **46** (1984) 369–372.

Second largest known maximal cliques in $P(q^2)$

Given an odd prime power q, put $r(q) := \begin{cases} 1, & q \equiv 1(4); \\ 3, & q \equiv 3(4). \end{cases}$

In 1996, Baker et al. found [3] maximal cliques of size $\frac{q+r(q)}{2}$ in $P(q^2)$ for any odd prime power q.

In 2018, Goryainov et al. found [4] one more family of maximal cliques in $P(q^2)$ with the same size $\frac{q+r(q)}{2}$.

[3] R. D. Baker, G. L. Ebert, J. Hemmeter, A. J. Woldar, *Maximal cliques in the Paley graph of square order*, J. Statist. Plann. Inference **56** (1996) 33–38.

[4] S. V. Goryainov, V. V. Kabanov, L. V. Shalaginov, A. A. Valyuzhenich, *On eigenfunctions and maximal cliques of Paley graphs of square order*, Finite Fields and Their Applications, **52**, (2018) 361–369.

Computations on maximal cliques of size $\frac{q+r(q)}{2}$ in $P(q^2)$

q	3	5	7	9	11	13	17	19	23
Clique size	3	3	5	5	7	7	9	11	13
#Orbits	1	1	1	3	3	4	9	4	4

q	25	27	29	31	37	41	43	47	49
Clique size	13	15	15	17	19	21	23	25	25
#Orbits	2	2	2	2	2	2	2	2	2

q	53	59	61	67	71	73	79	81	83
Clique size	27	31	31	35	37	37	41	41	43
#Orbits	2	2	2	2	2	2	2	2	2

Conjecture

For $q \geq 25$, the graph $P(q^2)$ contains exactly two non-equivalent cliques of size $\frac{q+r(q)}{2}$.

Finite field $\mathbb{F}_{q^2}$

Fix a non-square $d \in \mathbb{F}_q^*$.

Consider the polynomial $f(t) = t^2 - d \in \mathbb{F}_q[t]$.

Then

$$\mathbb{F}_{q^2} = \{x + y\alpha \mid x, y \in \mathbb{F}_q\},$$

where α is a root of $f(t)$.

Let β be a primitive element of $\mathbb{F}_{q^2}$.

Note that the elements from $\mathbb{F}_q^* = \langle \beta^{q+1} \rangle$ are squares in $\mathbb{F}_{q^2}^*$.

Affine plane $A(2, q)$

Let $V(2, q)$ be a 2-dimensional vector space over $\mathbb{F}_q$.

Consider the affine plane $A(2, q)$ whose

- points are vectors of $V(2, q)$;
- lines are all cosets of 1-dimensional subspaces in $V(2, q)$;
- incidence relation is natural (whether a vector belongs to a coset).

Since $\mathbb{F}_{q^2}$ can viewed as a 2-dimensional vector space over $\mathbb{F}_q$, the points of $A(2, q)$ can be matched with the elements of $\mathbb{F}_{q^2}$ as follows:

$$(x, y) \leftrightarrow x + y\alpha.$$

Quadratic and non-quadratic lines in $A(2, q)$

Given a line ℓ in $A(2, q)$, there exist elements $x_1 + y_1\alpha$ and $x_2 + y_2\alpha$ such that

$$\ell = \{x_1 + y_1\alpha + c(x_2 + y_2\alpha) \mid c \in \mathbb{F}_q\}.$$

The line ℓ is called quadratic (non-quadratic) if $x_2 + y_2\alpha$ is a square (non-square) in $\mathbb{F}^*_{q^2}$.

- The subfield $\mathbb{F}_q$ is a quadratic line.
- There are precisely $q + 1$ lines through a point: $\frac{q+1}{2}$ quadratic and $\frac{q+1}{2}$ non-quadratic lines.

$P(q^2)$ as a graph on points of the affine plane $A(2, q)$

For any distinct $\gamma_1, \gamma_2 \in \mathbb{F}_{q^2}$, the difference $\gamma_1 - \gamma_2$ is a square in $\mathbb{F}^*_{q^2}$ (equivalently, $\gamma_1 \sim \gamma_2$ in $P(q^2)$) iff the line connecting γ_1 and γ_2 is quadratic.

Take an element $\gamma \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$.

Since $\mathbb{F}_q$ is a quadratic line, the line through γ that is parallel to $\mathbb{F}_q$, is quadratic too.

The other $\frac{q-1}{2}$ quadratic lines through γ intersect $\mathbb{F}_q$ in $\frac{q-1}{2}$ points; denote this set of $\frac{q-1}{2}$ intersection points by X_γ.

For the conjugate element $\overline{\gamma}$, the equality $X_{\overline{\gamma}} = X_\gamma$ holds.

If $q \equiv 1(4)$, each of the sets $\{\gamma\} \cup X_\gamma$ and $\{\overline{\gamma}\} \cup X_\gamma$ induce a maximal clique of size $\frac{q+1}{2}$.

If $q \equiv 3(4)$, the set $\{\gamma, \overline{\gamma}\} \cup X_\gamma$ induces a maximal clique of size $\frac{q+3}{2}$.

[3] R. D. Baker, G. L. Ebert, J. Hemmeter, A. J. Woldar, *Maximal cliques in the Paley graph of square order*, J. Statist. Plann. Inference **56** (1996) 33–38.

The subgroup Q of order $q+1$ in $\mathbb{F}_{q^2}^*$

Put

$$\omega := \beta^{q-1}, Q := \langle \omega \rangle,$$

$$Q_0 := \langle \omega^2 \rangle, Q_1 := \omega \langle \omega^2 \rangle.$$

- Q is a subgroup of order $q+1$ in $\mathbb{F}_{q^2}^*$
- Q is the kernel of the norm mapping $N : \mathbb{F}_{q^2}^* \to \mathbb{F}^*$
- Q forms an oval in $A(2, q)$ (that is a set of $q+1$ points with no three on a line)
- Q is included to the neighbourhood of 0
- If $q \equiv 1(4)$, then Q induces the complete bipartite graph with parts Q_0 and Q_1
- If $q \equiv 3(4)$, then Q induces a pair of disjoint cliques Q_0 and Q_1

If $q \equiv 1(4)$, each of the sets Q_0 and Q_1 induces a maximal coclique of size $\frac{q+1}{2}$ in $P(q^2)$ (a maximal clique of size $\frac{q+1}{2}$ in $\overline{P(q^2)}$).

If $q \equiv 3(4)$, each of the sets $\{0\} \cup Q_0$ and $\{0\} \cup Q_1$ induces a maximal clique of size $\frac{q+3}{2}$ in $P(q^2)$.

[4] S. V. Goryainov, V. V. Kabanov, L. V. Shalaginov, A. A. Valyuzhenich, *On eigenfunctions and maximal cliques of Paley graphs of square order*, Finite Fields and Their Applications, **52**, (2018) 361–369.

Questions

Question 1.1

Is there any duality between the cliques from [3] and [4]?

Question 1.2

Is it true that the cliques from [3] and [4] are the only maximal cliques of size $\frac{q+r(q)}{2}$ in $P(q^2)$?

Question 1.3

Is it true that there are no maximal cliques whose size belongs to the gap from $\frac{q+r(q)}{2}$ to q?

Eigenfunction of a graph

Let $G = (V, E)$ be a regular graph.

Definition

A function $f : V \longrightarrow \mathbb{R}$ is called an eigenfunction of the graph G corresponding to an eigenvalue θ if $f \not\equiv 0$ and for any vertex γ the local condition

$$\theta \cdot f(\gamma) = \sum_{\delta \in G(\gamma)} f(\delta)$$

holds, where $G(\gamma)$ is the set of neighbours of the vertex γ.

A correspondence between eigenfunctions

Let $G = (V, E)$ be a regular graph.

A function f is an eigenfunction of the graph G corresponding to a non-principal eigenvalue θ iff f is an eigenfunction of the complementary graph $\overline{G}$ corresponding to the eigenvalue $-(1 + \theta)$.

Equivalent definition of eigenfunction

Let $G = (V, E)$ be a simple graph.

$V = \{v_1, v_2, \ldots, v_n\}$.

Eigenfunction of a graph(equivalent definition)

A function $f : V \longrightarrow \mathbb{R}$ is called an eigenfunction of G corresponding to θ if the vector $(f(v_1), \ldots, f(v_n))^T$ is an eigenvector of the adjacency matrix of G corresponding to the eigenvalue θ.

Strongly regular graphs

A graph G is called strongly regular with parameters (n, k, λ, μ), if G has n vertices, is regular of valency k, and any two vertices $\gamma, \delta \in V(G)$ have λ common neighbours, if $\gamma \sim \delta$, and μ common neighbours, if $\gamma \nsim \delta$.

- Non-trivial strongly regular graph with parameters (n, k, λ, μ) has exactly three eigenvalues: the principal eigenvalue k and the non-principal eigenvalues θ_1 and θ_2, which can be expressed in terms of the parameters n, k, λ, μ;
- The inequalities $\theta_2 < 0 < \theta_1 < k$ hold.

Eigenfunctions of Paley graphs with minimum cardinality of support

$P(q_1)$ is an SRG with parameters $(q_1, \frac{q_1-1}{2}, \frac{q_1-5}{4}, \frac{q_1-1}{4})$;

$P(q_1)$ has the non-principal eigenvalues $\theta_1 = \frac{-1+\sqrt{q_1}}{2}$ and $\theta_2 = \frac{-1-\sqrt{q_1}}{2}$;

Definition

For a function $f : V \to \mathbb{R}$, the set $Supp(f) := \{\gamma \in V \mid f(\gamma) \neq 0)\}$ is called the support of f.

Problem 1.2

To find the minimum cardinality of the support of eigenfunctions of Paley graphs corresponding to non-principal eigenvalues.

Problem 1.3

For Paley graphs, to characterize eigenfunctions with the minimum cardinality of support corresponding to non-principal eigenvalues.

An eigenfunction of $P(q^2)$ given by Q

Let $f_Q : \mathbb{F}_{q^2} \longrightarrow \mathbb{R}$ be a function defined by the following rule:

$$f_Q(\gamma) := \begin{cases} 1, & \text{if } \gamma \in Q_0; \\ -1, & \text{if } \gamma \in Q_1; \\ 0, & \text{otherwise.} \end{cases}.$$

Note that $|Supp(f_Q)| = q + 1$ holds.

If $q \equiv 1(4)$, then f_Q is an eigenfunction of $P(q^2)$ (of $\overline{P(q^2)}$) corresponding to the eigenvalue $\theta_2 = \frac{-1-q}{2}$ (to the eigenvalue $\theta_1 = \frac{-1+q}{2}$, respectively);

If $q \equiv 3(4)$, then f_Q is an eigenfunction of $P(q^2)$ (of $\overline{P(q^2)}$) corresponding to the eigenvalue $\theta_1 = \frac{-1+q}{2}$ (to the eigenvalue $\theta_2 = \frac{-1-q}{2}$, respectively).

Weight-distribution bound applied to $P(q^2)$

The weight-distribution bound (see [5]) says that an eigenfunction of $P(q^2)$ corresponding to a non-principal eigenvalue has at least $q+1$ non-zeroes.

Thus, the eigenfunction f_Q meets the weight-distribution bound.

Question 1.4

Is it true that every eigenfunction of $P(q^2)$ corresponding to a non-principal eigenvalue and having cardinality of support q+1, is equivalent to the eigenfunction f_Q?

[5] D. Krotov, I. Mogilnykh, V. Potapov, *To the theory of q-ary Steiner and other-type trades*, Discrete Mathematics, 2016. V. 339, N 3. pp. 1150–1157.

Result 2. Outline

1. Preliminary
2. Equitable partitions of Latin-square graphs
3. Further investigation

Equitable partitions

An equitable t-partition of a k-regular graph $\Gamma = (V, E)$ is a partition of the vertex set V into t parts $V_1, \ldots, V_t$ such that, for all $i, j \in \{1, ..., t\}$, every vertex of V_i is adjacent to the same number, namely, p_{ij}, of vertices of V_j. The matrix $P := (p_{ij})_{i,j=1,...,t}$ is called the quotient matrix of the equitable t-partition.

It is well known that every eigenvalue of P is an eigenvalue of the adjacency matrix of Γ.

In particular, P always has the principal eigenvalue k.

Let θ be a non-principal eigenvalue of Γ. We say that the partition P is θ-equitable if all non-principal eigenvalues of P are equal to θ.

Any equitable 2-partition is θ-equitable for some θ.

Perfect sets

We call a nonempty proper subset S of the vertex set V a θ-perfect set if the partition $\{S, V \setminus S\}$ is θ-equitable.

Lemma 2.1

A partition $\{V_1, \ldots, V_r\}$ is θ-perfect iff each set V_i is θ-perfect.

The following lemma can be found in [6].

Lemma 2.2

Let S be a θ-perfect set, and T a non-empty proper subset of $V \setminus S$. Then T is θ-perfect iff $S \cup T$ is θ-perfect.

Thus, to find all θ-equitable partitions, it is sufficient to find all the minimal θ-perfect sets.

[6] D. Krotov, *On perfect colorings of the halved 24-cube*, Diskretnyi Analiz i Issledovanie Operatsii 15 (2008), 35–46. (In Russian; English translation at http://arxiv.org/abs/0803.0068.)

Latin square graphs

A Latin square of order n is an $n \times n$ array with entries from an alphabet of n letters, such that each letter occurs once in each row and once in each column.

Given a Latin square L, we define the corresponding Latin square graph $\Gamma(L)$ whose vertices are the n^2 cells of the array L, two vertices are adjacent iff they lie in the same row or the same column or contain the same letter.

The graph $L(n)$ is strongly regular with eigenvalues $k = 3(n-1)$, $\theta_1 = n-3$ and $\theta_2 = -3$.

$(n-3)$-perfect sets given by a clique

Lemma 2.3

Let S be a row, a column, or a letter. Then S is a $(n-3)$-perfect set.

$(n-3)$-perfect sets given by a corner

0	1	2	3	4
1	2	3	4	0
2	3	4	0	1
3	4	0	1	2
4	0	1	2	3

Lemma 2.3

Let L be the Cayley table of a cyclic group and S be the left upper corner under the secondary diagonal. Then S is a $(n-3)$-perfect set.

Inflation of Latin squares

Take a Latin square L_0 of order s.

Replace each occurrence of letter i by a Latin square of order t in alphabet A_i, where the alphabets for different letters are pairwise disjoint; this gives a Latin square L of order $n = st$.

Moreover, given an $(s-3)$-perfect set S_0 in L_0, the corresponding cells in L form an $(n-3)$-perfect set.

The theorem

The following theorem exhaust (see [7]) the minimal $(n-3)$-perfect sets.

Theorem 2.1

Let S be a minimal $(n-3)$-perfect set in the graph of a Latin square of order n. Then S is a row, a column, a letter, or an inflation of a corner set.

0	1	2
1	2	0
2	0	1

$\rightsquigarrow$

3	4	5	6	7	8
4	3	6	5	8	7
6	5	7	8	4	3
5	6	8	7	3	4
8	7	3	4	6	5
7	8	4	3	5	6

[7] R. A. Bailey, P. J. Cameron, A. L. Gavrilyuk and S. V. Goryainov, *Equitable partitions of Latin-square graphs*, February 2018, arXiv:1802.01001, accepted to Journal of Combinatorial Designs.

Questions

Question 2.1

Studying -3-perfect sets. Such a set S has the property that it meets any row, column or letter in a constant number s of cells, and its cardinality is sn. In particular, with $s = 1$, such set is transversal. A classification of (-3)-perfect sets would imply a solution for the long standing Ryser's conjecture.

Question 2.2

Some families of distance-regular graphs have nonempty intersection with the family of Latin-square graphs (for example, bilinear forms graph). Can we classify their equitable partitions?

Question 2.3

Can we classify equitable partition of graphs of mutual orthogonal Latin squares?

Result 3. Outline

1. Preliminary
2. Two extensions of the smallest strictly Neumaier graph
3. Open questions

Affine polar graph

Let V be a $(2e)$-dimensional vector space over a finite field $\mathbb{F}_q$, where $e \geq 2$ and q is a prime power, provided with the hyperbolic quadratic form $Q(x) = x_1x_2 + x_3x_4 + \ldots + x_{2e-1}x_{2e}$.

The set Q^+ of zeroes of Q is called the hyperbolic quadric, where e is the maximal dimension of a subspace in Q^+. A generator of Q^+ is a subspace of maximal dimension e in Q^+.

Denote by $VO^+(2e, q)$ the graph on V with two vectors x, y being adjacent iff $Q(x - y) = 0$.

Lemma

A graph $VO^+(2e, q)$ is a vertex transitive strongly regular graph with parameters

$$v = q^{2e}, k = (q^{e-1} + 1)(q^e - 1),$$

$$\lambda = q(q^{e-2} + 1)(q^{e-1} - 1) + q - 2, \mu = q^{e-1}(q^{e-1} + 1).$$

Affine polar graph

Note that $VO^+(2e, q)$ is isomorphic to the graph defined on the set of all $(2 \times e)$-matrices over $\mathbb{F}_q$

$$\{\begin{pmatrix} x_1 & x_3 & \dots & x_{2e-1} \\ x_2 & x_4 & \dots & x_{2e} \end{pmatrix}\},$$

where two matrices are adjacent iff the scalar product of the first and the second rows of their difference is equal to 0.

A spread in $VO^+(2e, q)$ is a set of q^e disjoint maximal cliques that correspond to all cosets of a generator.

We have found [8] two generalisations of the smallest striclty Neumaier. Thus, there is a strictly Neumaier graph on 2^{2e} vertices containing a 2^{e-1}-regular 2^e-clique.

[8] R. J. Evans, S. V. Goryainov and D. I. Panasenko, *The smallest strictly Neumaier graph and its generalisations*, in preparation.

Questions

Question 3.1

Is there a strictly Neumaier graph with nexus that is not a power of 2?

Question 3.2

Is there a strictly Neumaier graph of diameter 3?

Question 3.3

How many non-isomorphic strictly Neumaier graphs can we produce iterating the construction?

Thank you for your attention!