Extremal Peisert-type graphs without strict-EKR property

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Erdős–Ko–Rado theorem

The Erdős–Ko–Rado theorem, one of the fundamental results in combinatorics, provides information about systems of intersecting sets. A family $\mathcal{A}$ of subsets of a ground set — it might as well be $\{1, \ldots, n\}$ — is intersecting if any two sets in $\mathcal{A}$ have at least one point in common.

Theorem 1 (Erdős–Ko–Rado, 1961)

Let k and n be integers with $n \geq 2k$. If $\mathcal{A}$ is an intersecting family of k-subsets of $\{1, \ldots, n\}$, then

$$|\mathcal{A}| \leq \binom{n-1}{k-1}.$$

Moreover, if $n > 2k$, equality holds if and only if $\mathcal{A}$ consists of all the k-subsets that contain a given point from $\{1, \ldots, n\}$.

Extensions of Erdős–Ko–Rado theorem

This theorem has two parts: a bound and a characterisation of families that meet the bound.

One reason this theorem is so important is that it has many interesting extensions. In particular, it can be translated to a question in graph theory. The Kneser graph $K(n, k)$ has all k-subsets of $\{1, \ldots, n\}$ as its vertices, and two k-subsets are adjacent if they are disjoint. (We assume $n \geq 2k$ to avoid trivialities.)

Then an intersecting family of k-subsets is a coclique in the Kneser graph, and we see that the EKR theorem characterises the cocliques of maximum size in the Kneser graph. An intersecting family consisting of all subsets containing a given point is called canonical.

So we can seek to extend the EKR theorem by replacing the Kneser graphs by other interesting families of graphs.

EKR properties

Given any graph X for which we can describe its canonical cliques (that is, typically cliques with large size and simple structure; in best case, cliques are canonical with respect to some notion of 'intersecting' defined for the vertices), we can ask whether X has any of the following three related Erdős-Ko-Rado (EKR) properties:

- EKR property: the clique number of X equals the size of canonical cliques.
- EKR-module property: the characteristic vector of each maximum clique in X is a $\mathbb{Q}$-linear combination of characteristic vectors of canonical cliques in X.
- strict-EKR property: each maximum clique in X is a canonical clique.

For a graph with EKR-property and without strict-EKR property, we say that a maximum clique is non-canonical if it is not canonical.

EKR-type results

The classical Erdős-Ko-Rado theorem [EKR61] classified maximum intersecting families of k-element subsets of $\{1, 2, \ldots, n\}$ when $n \geq 2k + 1$.

Since then, EKR-type results refer to understanding maximum intersecting families in a broader context, and more generally, classifying extremal configurations in other domains. The book [GM15] by Godsil and Meagher provides an excellent survey on the modern algebraic approaches to proving EKR-type results for permutations, set systems, orthogonal arrays, and so on.

[EKR61] P. Erdős, C. Ko, and R. Rado, Intersection theorems for systems of finite sets, Quart. J. Math. Oxford Ser. (2) 12 (1961), 313–320.

[GM15] C. D. Godsil, K. Meagher, *Erdős-Ko-Rado Theorems: Algebraic Approaches*, Cambridge University Press (2015).

Block graphs of orthogonal arrays

In this talk, we consider the definition of the notion of 'intersecting' for the vertices of the block graphs of orthogonal arrays (that is, for columns of an orthogonal array). Once we have done this, we can define the notion of canonical intersecting families in these graphs.

An orthogonal array $OA(m, n)$ is an $m \times n^2$ array with entries from an n-element set T with the property that the columns of any $2 \times n^2$ subarray consist of all n^2 possible pairs.

The block graph of an orthogonal array $OA(m, n)$, denoted $X_{OA(m,n)}$, is defined to be the graph whose vertices are columns of the orthogonal array, where two columns are adjacent if there exists a row in which they have the same entry.

Two columns of an orthogonal array are called intersecting if they have the same entry in some row.

Canonical cliques in the block graphs of orthogonal arrays

Let $S_{r,i}$ be the set of columns of $OA(m,n)$ that have the entry i in row r. These sets are cliques, and since each element of the n-element set T occurs exactly n times in each row, the size of $S_{r,i}$ is n for all i and r. These cliques are called the canonical cliques in the block graph $X_{OA(m,n)}$.

A simple combinatorial argument shows that the block graph of an orthogonal array is strongly regular. Moreover, by the Delsarte bound, a clique in $X_{OA(m,n)}$ has size at most n, and the canonical cliques show the tightness of this bound.

Intersecting columns in an orthogonal array

If we view columns of an orthogonal array that have the same entry in the same row as intersecting columns, then we can view the Delsarte bound as the bound in the EKR theorem for intersecting columns of an orthogonal array. The question is, under what conditions will all cliques of size n in the graph $X_{OA}(m,n)$ be canonical? The following answer can be viewed as the uniqueness part of the EKR theorem.

Theorem 2 ([GM15, Corollary 5.5.3])

Let $X = X_{OA(m,n)}$ be the block graph of an orthogonal array $OA(m,n)$ with $n > (m-1)^2$ (equivalently, $m < \sqrt{n}+1$). Then X has the strict-EKR property: the only maximum cliques in X are the columns that have entry i in row r for some $1 \le i \le n$ and $1 \le r \le m$.

This is equivalent to saying that the largest set of intersecting columns in an orthogonal array is the set of all columns that have the same entry in the some row, and these sets are the only maximum intersecting sets.

Open problems for the block graphs of orthogonal arrays

Problem 1

Find a characterisation of the orthogonal arrays, based only on the parameters of the array, for which all of the maximum cliques in the orthogonal array graph are canonical cliques.

Problem 2

Assume that $OA(m, (m-1)^2)$ *is an orthogonal array and its orthogonal array graph has non-canonical cliques of size* $(m-1)^2$*. Do these non-canonical cliques form subarrays?*

[GM15, Section 5.5] provides an example of non-canonical cliques in the block graph of an orthogonal array that form subarrays. In this talk, we will meet a subfamily of affine polar graphs that can be viewed as block graphs of orthogonal arrays containing non-canonical cliques having subarray structure.

[GM15] C. D. Godsil, K. Meagher, *Erdös-Ko-Rado Theorems: Algebraic Approaches*, Cambridge University Press (2015).

A general problem for block graphs of orthogonal arrays

Problem 3

Determine all the maximum cliques in the block graph of an arbitrary orthogonal array.

Affine plane $\mathrm{AG}(2, q)$

Let q be a prime power and W be a 2-dimensional vector space over the finite fields $GF(q)$.

Then the set of all cosets of 0-dimensional and 1-dimensional subspaces of W, ordered by inclusion, forms an affine plane of order q, denoted by $\mathrm{AG}(2, q)$.

Identification of the elements of $\mathbb{F}_{q^2}$ and the points of AG$(2, q)$

Let $F = \mathbb{F}_{q^2}$ for some prime power q. Then F can be viewed in a canonical way as a two-dimensional vector space over $\mathbb{F}_q$, or, as the affine plane $AG(2, q)$.

Each nonzero element uniquely defines a line through 0 and can be viewed as a slope (direction) of this line.

Peisert-type graphs

Let q be a prime power. Let $S \subset \mathbb{F}_{q^2}^*$ be a union of $m \leq q$ cosets of $\mathbb{F}_q^*$ in $\mathbb{F}_{q^2}^*$ such that $\mathbb{F}_q^* \subset S$, that is,

$$S = c_1\mathbb{F}_q^* \cup c_2\mathbb{F}_q^* \cup \cdots \cup c_m\mathbb{F}_q^*.$$

Then the Cayley graph $X = \mathrm{Cay}(\mathbb{F}_{q^2}^+, S)$ is said to be a Peisert-type graph of type (m, q). A clique in X is called a canonical clique if it is the image of the subfield $\mathbb{F}_q$ under an affine transformation.

Peisert-type graphs are equivalent to the block graphs of orthogonal arrays obtained from the parallel classes of the affine plane $\mathrm{AG}(2, q)$.

Another way to look at Peisert-type graphs is to consider them as fusions of certain amorphic cyclotomic association schemes.

Basic properties of Peisert-type graphs

- Every Peisert-type graph of type (m, q) can be naturally defined on the points of the affine plane $\mathrm{AG}(2, q)$ with two points being adjacent whenever the line through these points belongs to one of m prescribed parallel classes of lines; the canonical cliques in a Peisert-type graph of type (m, q) are exactly the lines from m prescribed parallel classes defining the graph.
- The affine plane $\mathrm{AG}(2, q)$ can be viewed as an orthogonal $(q + 1) \times q^2$-array $OA(q + 1, q)$; every Peisert-type of type (m, q) graph can be viewed as the block graph of an orthogonal array $OA(m, q)$ obtained from this array $OA(q + 1, q)$ by choosing the subset of m rows corresponding the m prescribed classes of parallel classes.
- The definitions of canonical cliques in the block graphs of orthogonal arrays and Peisert-type graphs agree with each other.

Intersections of the class of Peisert-type graphs with some other classes

- Paley graphs $P(q^2)$ of square order are Peisert-type graphs;
- Peisert graphs $P^*(q^2)$, where $q \equiv 3 \pmod 4$, are Peisert-type graphs (not all Peisert graphs are Peisert-type graphs);
- Generalised Paley graphs $GP(q^2, d)$, where $d \mid (q+1)$ and $d > 1$ (not all generalised Paley graphs are Peisert-type graphs);
- Generalised Peisert graphs $GP^*(q^2, d)$, where $d \mid (q+1)$ and d is even (not all generalised Peisert graphs are Peisert-type graphs).

Balanced characteristic vectors

A vector in $\mathbb{R}^n$ is balanced if it is orthogonal to the all-ones vector $\mathbf{1}$. If v_S is the characteristic vector of a subset S of the set V, then we say that

$$v_S - \frac{|S|}{|V|}\mathbf{1}$$

is the balanced characteristic vector of S.

In [B84], Blokhuis proved that Paley graphs of square order have the strict-EKR property.

Any clique of size q in the Paley graph $P(q^2)$ is a Delsarte clique, and so its balanced characteristic vector lies in the $\frac{-1+q}{2}$-eigenspace of the Paley graph. In [GM15, Section 5.9], a possible alternate proof of Blokhuis's result is presented; this proof relies on the following problems.

[B84] A. Blokhuis, On subsets of $GF(q^2)$ with square differences, Indag. Math. 46 (1984) 369–372.

A question on Paley graphs of square order by Godsil and Meagher

Problem 4

Show that the $\frac{-1+q}{2}$-eigenspace of $P(q^2)$ is spanned by the balanced characteristic vectors of the canonical cliques.

In [AGLY22], we showed that Problem 4 is equivalent to establishing the EKR-module property for Paley graphs of square order. Moreover, we established the EKR-module property for all block graphs of orthogonal arrays (including Peisert-type graphs) and thus solved Problem 4.

Problem 5

Prove that the only balanced characteristic vectors of sets of size q, in the $\frac{-1+q}{2}$-eigenspace of $P(q^2)$, are the balanced characteristic vectors of the canonical cliques.

[AGLY22] S. Asgarli, S. Goryainov, H. Lin, C. H. Yip, *The EKR-module property of pseudo-Paley graphs of square order*, Electron. J. Combin. **29** (2022), no. 4, #P4.33.

Module method

In general, the module method (see [AM15, Section 4]) refers to the strategy of proving that a graph X satisfies the strict-EKR property in two steps:

- show that X satisfies the EKR-module property
- show that EKR-module property implies the strict-EKR property

As an example of the module method, [AM15, Theorem 4.5] provides a sufficient condition for the second step above for 2-transitive permutation groups.

[AM15] B. Ahmadi and K. Meagher, *The Erdős-Ko-Rado property for some 2-transitive groups*, Ann. Comb. 19 (2015), no. 4, 621–640.

Further, we list another recent results related to EKR-properties of Peisert-type graphs

Subspace structure of Delsarte cliques in Peisert-type graphs

The following theorem shows that all maximum cliques (canonical and non-canonical (if any)) in Peisert-type graphs of type (m, q), where $m \leq \frac{q+1}{2}$ have nice algebraic structure.

Theorem 3 ([AY22, Theorem 1.2])

Let X be a Peisert-type graph of type (m, q), where q is a power of an odd prime p and $m \leq \frac{q+1}{2}$. Then any maximum clique in X containing 0 is an $\mathbb{F}_p$-subspace of $\mathbb{F}_{q^2}$.

It is a good open problem to give symmetric constructions of Peisert-type graphs with $m > \frac{q+1}{2}$ and their non-canonical cliques without a subspace structure.

[AY22] S. Asgarli, C. H. Yip, *Van Lint-MacWilliams' conjecture and maximum cliques in Cayley graphs over finite fields*, J. Combin. Theory Ser. A **192** (2022), Paper No. 105667, 23 pp.
`https://doi.org/10.1016/j.jcta.2022.105667`

Erdős–Ko–Rado theorem in Peisert-type graphs (I)

The stability of canonical cliques in Peisert-type graphs has also been studied. For simplicity, we say a Peisert-type graph X with order q^2 has the property $ST(k)$ if each clique in X with size at least $q-k$ is contained in a canonical clique in X. From the definition, it is clear that the strict-EKR property is equivalent to the property $ST(0)$.

Theorem 4 ([Y23])

As $q \to \infty$, where q is a prime power, almost all Peisert-type graphs of type $(\frac{q+1}{2}, q)$ have the property $ST(\frac{\sqrt{q}}{2})$.

Theorem 5 ([Y23])

As $p \to \infty$, where p is prime, almost all Peisert-type graphs of type $(\lfloor \frac{2p-2}{3} \rfloor, p)$ have the strict-EKR property, and almost all Peisert-type graphs of type $(\frac{p+1}{2}, p)$ have the property $ST(p/20)$.

[Y23] C. H. Yip, *Erdős–Ko–Rado theorem in Peisert-type graphs*, https://arxiv.org/abs/2302.00745

Erdős–Ko–Rado theorem in Peisert-type graphs (II)

Theorem 6 ([Y23])

Let $n \geq 2$ with the largest proper divisor being t and let $q = p^n$. As $p \to \infty$, almost all Peisert-type graphs of type $(q - o(p^t), q)$ do not have the strict-EKR property.

[Y23] C. H. Yip, *Erdős–Ko–Rado theorem in Peisert-type graphs*, `https://arxiv.org/abs/2302.00745`

From now, we present the main results of this talk.

Number of pairwise non-isomorphic Peisert-type graphs

It is well-known that the stabiliser of the zero point of $\mathrm{AG}(2,q)$ acts 3-transitively on the set of lines through this point; this implies that, for any prime power q, all Peisert-type graphs of type $(3,q)$ are isomorphic.

Since the complement of a Peisert-type graph of type (m,q) is a Peisert-type graph of type $(q+1-m,q)$, we conclude that the number of pairwise non-isomorphic Peisert-type graphs of type (m,q) is equal to the number of pairwise non-isomorphic Peisert-type graphs of type $(q+1-m,q)$.

Whenever $q \leq 5$, there exists a unique Peisert-type graph of type (m,q) for any admissible value of m, that is, $1 \leq m \leq q$.

Extremal Peisert-type graphs without strict-EKR property

Theorem 7 ([AGLY22])

If $q > (m-1)^2$ (equivalently, $m < \sqrt{q}+1$), then all Peisert-type graphs of type (m, q) have the strict-EKR property. Moreover, when q is a square, there exists a Peisert-type graph of type $(\sqrt{q}+1, q)$ without the strict-EKR property.

Given a prime power q, there exists the smallest value of m, say e_q, such that there exists a Peisert-type graph without strict-EKR property; we call such parameter e_q extremal.

In [AGLY22], we showed (in a non-constructive way) that if q is a square, then $e_q = \sqrt{q}+1$. In general, we have $e_q \geq \sqrt{q}+1$.

A Peisert-type graph of type (e_q, q) without strict-EKR property is called extremal.

[AGLY22] S. Asgarli, S. Goryainov, H. Lin, C. H. Yip, *The EKR-module property of pseudo-Paley graphs of square order*, Electron. J. Combin. **29** (2022), no. 4, #P4.33.

Number of pairwise non-isomorphic Peisert-type graphs

$q = 4:$

m	3	4
#Graphs	1	1
strict-EKR	-	-
without	1	1

$q = 5:$

m	3	4	5
#Graphs	1	1	1
strict-EKR	1	-	-
without	-	1	1

$q = 7:$

m	3	4	5	6
#Graphs	1	2	1	1
strict-EKR	1	2	-	-
without	-	-	1	1

$q = 8:$

m	3	4	5	6
#Graphs	1	1	1	1
strict-EKR	1	1	-	-
without	-	-	1	1

We thus have $e_4 = 3$, $e_5 = 4$, $e_7 = 5$ and $e_8 = 5$. In these four cases, an extremal graph is unique.

Number of pairwise non-isomorphic Peisert-type graphs

$q = 9:$

m	3	4	5	6	7
#Graphs	1	2	2	2	1
strict-EKR	1	1	1	-	-
without	-	1	1	2	1

$q = 11:$

m	3	4	5	6	7	8	9
#Graphs	1	2	2	4	2	2	1
strict-EKR	1	2	2	4	1	1	-
without	-	-	-	-	1	1	1

We thus have $e_9 = 4$, $e_{11} = 7$. In these two cases, an extremal graph is unique.

Number of pairwise non-isomorphic Peisert-type graphs

$q = 13:$

m	3	4	5	6	7	8	9	10	11
#Graphs	1	3	3	5	5	5	3	3	1
strict-EKR	1	3	3	5	5	4	2	-	-
without	-	-	-	-	-	1	1	3	1

$q = 16:$

m	3	4	5	6	7	8	9	10	11	12	13	14
#Graphs	1	2	3	4	5	6	6	5	4	3	2	1
strict-EKR	1	2	2	3	3	3	1	-	-	-	-	-
without	-	-	1	1	2	3	5	5	4	3	2	1

We thus have $e_{13} = 8$, $e_{16} = 5$. In these two cases, an extremal graph is unique.

Number of pairwise non-isomorphic Peisert-type graphs

$q = 17$:

m	3	4	5	6	7	8	9	10	11	12	13	14	15
#Graphs	1	3	4	10	10	17	17	17	10	10	4	3	1
strict-EKR	1	3	4	10	10	17	17	16	9	5	1	-	-
without	-	-	-	-	-	-	-	1	1	5	3	3	1

We thus have $e_{17} = 10$, and an extremal graph is unique.

Number of pairwise non-isomorphic Peisert-type graphs

$q = 19:$

m	3	4	5	6	7	8	9	10	11	12	13	14
#Graphs	1	4	5	13	18	31	33	44	33	31	18	13
strict-EKR	1	4	5	13	18	31	33	44	32	30	14	5
without	-	-	-	-	-	-	-	-	1	1	4	8

m	15	16	17
#Graphs	5	4	1
strict-EKR	-	-	-
without	5	4	1

We thus have $e_{19} = 11$, and an extremal graph is unique.

Number of pairwise non-isomorphic Peisert-type graphs

$$q = 23:$$

m	3	4	5	6	7	8	9	10	11	12	13
#Graphs	1	4	6	22	36	83	125	196	227	268	227
strict-EKR	1	4	6	22	36	83	125	196	227	268	226
without	-	-	-	-	-	-	-	-	-	-	1

m	14	15	16	17	18	19	20	21
#Graphs	196	125	83	36	22	6	4	1
strict-EKR	195	120	73	19	≥ 1	-	-	-
without	1	5	10	17	≥ 1	6	4	1

We thus have $e_{23} = 13$, and an extremal graph is unique.

Number of pairwise non-isomorphic Peisert-type graphs

$$q = 25:$$

m	3	4	5	6	7	8	9	10	11	12	13
#Graphs	1	4	7	19	34	79	132	223	293	379	391
strict-EKR	1	4	7	18	33	75	121	185	208	198	108
without	-	-	-	1	1	4	11	38	85	181	283

m	14	15	16	17	18	19	20	21	22	23
#Graphs	379	293	223	132	79	34	19	7	4	1
strict-EKR	34	3	1	-	-	-	-	-	-	-
without	345	290	222	132	79	34	19	7	4	1

We thus have $e_{25} = 6$, and an extremal graph is unique.

Main problem

This talk is mainly devoted to the following problem.

Problem 6

Determine all extremal Peisert-type graphs without strict-EKR property.

Main results (I)

Theorem 8 ([GY23, Theorem 1.3])

Let $q = p^n$, where p is a prime and n is a positive integer. Let X be an extremal Peisert-type graph defined over $\mathbb{F}_{q^2}$.

- *If $n = 1$ and $p \geq 3$, then X is of type $(\frac{p+3}{2}, p)$ and X is unique up to isomorphism.*
- *If $n > 1$, then X is of type $(p^{n-k} + 1, q)$, where k is the largest proper divisor of n.*

Note that the result in the case $n = 1$ follows from some known results from the theory of directions in affine planes, and the case $n > 1$ was newly developed.

We also note that, in the case $n > 1$, for any prime power q, we constructed an explicit example of an extremal Peisert-type graph without strict-EKR property (of type $(p^{n-k} + 1, q)$, where k is the largest proper divisor of n).

[GY23] S. Goryainov, C. H. Yip, *Extremal Peisert-type graphs without the strict-EKR property*, June 2023, `https://arxiv.org/abs/2206.15341`

Main results (II)

Theorem 9 ([GY23, Theorem 1.4])

Let q be the square of a prime power. There are exactly $(q+1)\sqrt{q}$ extremal Peisert-type graph defined over $\mathbb{F}_{q^2}$ and each Peisert-type graph of type $(3,q)$ is a subgraph of exactly one such extremal graph. Moreover, if X is such an extremal graph, then the following statements hold:

- *X is unique up to isomorphism: in fact, X is isomorphic to the affine polar graph $VO^+(4,\sqrt{q})$.*
- *X has exactly $\sqrt{q}+1$ canonical cliques containing 0, and $\sqrt{q}+1$ non-canonical cliques containing 0; moreover, these $2(\sqrt{q}+1)$ cliques lie in the same orbit under the action of the automorphism group of X.*
- *There is no Hilton-Milner type result: all maximal cliques in X are maximum cliques.*
- *The weight-distribution bound is tight for both non-principal eigenvalues of X.*

Main results (III)

Theorem 10 ([GY23, Theorem 1.5])

Let $q = r^3$, where r is a prime power and a non-square. There are exactly $r(r^5 + r^4 + r^3 + r^2 + r + 1)$ extremal Peisert-type graphs defined over $\mathbb{F}_{q^2}$. Moreover, if X is such an extremal graph, then the following statements hold:

- *X is unique up to isomorphism.*
- *Maximum cliques in X can be explicitly determined. In particular, X has exactly $r^2 + 1$ canonical cliques containing 0, and $r^2 + r + 1$ non-canonical cliques containing 0.*

[GY23] S. Goryainov, C. H. Yip, *Extremal Peisert-type graphs without the strict-EKR property*, June 2023, `https://arxiv.org/abs/2206.15341`

Main results (IV)

Let $q = 2^5$. Let ε be a primitive element in $\mathbb{F}_q$ such that ε is a root of the irreducible polynomial $t^5 + t^2 + 1 \in \mathbb{F}_2[t]$. Let β be a root of the irreducible polynomial $t^2 + t + 1 \in \mathbb{F}_2[t]$; note that $\beta \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$. Consider the Peisert-type graph X_1 induced by the $\mathbb{F}_2$-subspace V_1 generated by the elements $\{1, \varepsilon, \beta, \varepsilon^{16}\beta, \varepsilon^{21} + \varepsilon^9\beta\}$, that is, $X_1 = \mathrm{Cay}(\mathbb{F}_{q^2}^+, V_1(\mathbb{F}_q \setminus \{0\}))$. Similarly, consider the Peisert-type graph X_2 induced by the $\mathbb{F}_2$-subspace V_2 generated by the elements $\{1, \varepsilon, \varepsilon^2, \varepsilon^3, \beta\}$.

For $i \in \{1, 2\}$, it is easy to verify that V_i is a non-canonical clique in X_i and X_i is a Peisert-type graph of type $(17, 32)$. Thus, in view of Theorem 8, X_1 and X_2 are extremal graphs. We have verified that X_1 is not isomorphic to X_2, showing that there are at least 2 non-isomorphic extremal Peisert-type graphs defined over $\mathbb{F}_{q^2}$.

The set of directions of a subsets of points of an affine plane

Let U be a subset of $\mathrm{AG}(2,q)$; the set of directions determined by U is defined to be

$$D(U) := \{[a-c : b-d] : (a,b),(c,d) \in U,\ (a,b) \neq (c,d)\} \subset \mathrm{PG}(1,q).$$

The theory of directions has been developed by Rédei [R73], Szőnyi [S99], and many other authors. It is of particular interest to estimate $|D(U)|$.

[R73] L. Rédei, *Lacunary polynomials over finite fields*, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973. Translated from the German by I. Földes.

[S99] T. Szőnyi, *On the number of directions determined by a set of points in an affine Galois plane*, J. Combin. Theory Ser. A, 74(1):141–146, 1996.

Summary of the results on the size of $D(U)$; q is prime

Theorem 11

Let U be a subset of $\mathrm{AG}(2,p)$ *with* $|U| = p$*.*

- *(Rédei* [R73]*). If the points in U are not all collinear, then U determines at least $\frac{p+3}{2}$ directions.*
- *(Lovász and Schrijver* [LS83]*) If U determines exactly $(p+3)/2$ directions, then U is affinely equivalent to the set* $\{(x, x^{(p+1)/2}) : x \in \mathbb{F}_p\}$.
- *(Gács* [G03]*) If U determines more than $\frac{p+3}{2}$ directions, then it determines at least $\lfloor \frac{2p+1}{3} \rfloor$ directions.*

[R73] L. Rédei, *Lacunary polynomials over finite fields*, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973. Translated from the German by I. Földes.

[LS83] L. Lovász, A. Schrijver, *Remarks on a theorem of Rédei*, Studia Sci. Math. Hungar., 16 (1983) 449–454.

[G03] A. Gács, *On a generalization of Rédei's theorem.* Combinatorica, 23(4):585–598, 2003.

Implications to the theory of directions

Our main results imply an analogue of the result from Theorem 11 when q is a square or a cube (instead of a prime).

In other words, our main results strengthen the results due to Blokhuis, Ball, Brouwer, Storme, and Szőnyi [B03], [BBBSS99] in the sense that we classify non-collinear sets $U \subset AG(2, q)$ with q points that determines the minimum number of directions when q is a square or a cube, as well as such direction sets.

[BBBSS99] A. Blokhuis, S. Ball, A. E. Brouwer, L. Storme, and T. Szőnyi, *On the number of slopes of the graph of a function defined on a finite field*, J. Combin. Theory Ser. A, **86** (1999), no.1, 187–196.
`https://doi.org/10.1006/jcta.1998.2915`

[B03] S. Ball, *The number of directions determined by a function over a finite field*, J. Combin. Theory Ser. A **104** (2003), no. 2, 341–350.
`https://doi.org/10.1016/j.jcta.2003.09.006`

Concluding remarks

In a similar manner, given a prime power q, there exists the largest value of m, say E_q, such that there exists a Peisert-type graph of type (m, q) with strict-EKR property; one can also call such parameter E_q extremal.

A Peisert-type graph of type (E_q, q) with strict-EKR property is called extremal.

Problem 7

Given a prime power q, determine the value of E_q and characterise extremal Peisert-type graphs with strict-EKR property.

From the tables above, we know the numbers E_q for all $q \leq 25$. Moreover, when q is a small square, that is, $q \in \{4, 9, 16, 25\}$, an extremal graph with strict-EKR property is unique.

We plan to work on determining the value of E_q and investigating whether a uniquness result holds for extremal Peisert-type graphs with strict-EKR property.

Thank you for your attention!