Lecture 2: EKR properties of graphs

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Erdős–Ko–Rado theorem

The Erdős–Ko–Rado theorem, one of the fundamental results in combinatorics, provides information about systems of intersecting sets. A family $\mathcal{A}$ of subsets of a ground set — it might as well be $\{1, \ldots, n\}$ — is intersecting if any two sets in $\mathcal{A}$ have at least one point in common.

More generally a family $\mathcal{A}$ of subsets is t-intersecting if any two elements of $\mathcal{A}$ have at least t points in common.

Theorem 1 (Erdős–Ko–Rado, 1961)

Let k and n be integers with $n \geq 2k$. If $\mathcal{A}$ is an intersecting family of k-subsets of $\{1, \ldots, n\}$, then

$$|\mathcal{A}| \leq \binom{n-1}{k-1}.$$

Moreover, if $n > 2k$, equality holds if and only if $\mathcal{A}$ consists of all the k-subsets that contain a given point from $\{1, \ldots, n\}$.

Extensions of Erdős–Ko–Rado theorem (I)

This theorem has two parts: a bound and a characterisation of families that meet the bound.

One reason this theorem is so important is that it has many interesting extensions. To address these, we first translate it to a question in graph theory. The Kneser graph $K(n, k)$ has all k-subsets of $\{1, \ldots, n\}$ as its vertices, and two k-subsets are adjacent if they are disjoint. (We assume $n \geq 2k$ to avoid trivialities.)

Then an intersecting family of k-subsets is a coclique in the Kneser graph, and we see that the EKR theorem characterises the cocliques of maximum size in the Kneser graph.

So we can seek to extend the EKR theorem by replacing the Kneser graphs by other interesting families of graphs.

There is a second class of extensions of the EKR theorem. In their paper Erdős, Ko and Rado proved the following:

Extensions of Erdős–Ko–Rado theorem (II)

Theorem 2 (Erdős–Ko–Rado, 1961)

Let n, k and t be positive integers with $0 \leq t \leq k$. There exists a function $f(k,t)$ such that if $n \geq f(k,t)$ and $\mathcal{A}$ is a t-intersecting family of k-subsets of $\{1,\ldots,n\}$, then

$$|\mathcal{A}| \leq \binom{n-t}{k-t}.$$

Moreover, equality holds if and only if $\mathcal{A}$ consists of the k-subsets that contain a specified t-subset of $\{1,\ldots,n\}$.

There are graph-theoretic analogues of this question too. In place of the Kneser graphs, we use the Johnson graphs $J(n,k)$. The vertices of $J(n,k)$ are the k-subsets of $\{1,\ldots,n\}$, but now two k-subsets are adjacent if they have exactly $k-1$ points in common. Again we assume $n \geq 2k$. The graph $J(n,k)$ has diameter k and thus two k-subsets are adjacent in $K(n,k)$ if and only if they are at maximum possible distance in $J(n,k)$.

Extensions of Erdős–Ko–Rado theorem (III)

Define the width of a subset of the vertices of a graph to be the maximum distance between two vertices in the subset.

Then our first version of the EKR theorem characterises the subsets of maximum size in $J(n,k)$ of width $k-1$, and the second version the subsets of maximum size with width $k-t$.

All known analogues of the EKR theorem for t-intersecting sets can be stated naturally as characterisations of subsets of width $d-t$ in a graph of diameter d. However, such theorems have only been proved in cases where the distance graphs form an association scheme (they fit together in a particularly nice way).

To give one example, we can replace k-subsets of $\{1,\ldots,n\}$ by subspaces of dimension k over a vector space of dimension n over $GF(q)$.

Canonical t-intersecting families

It is easy to find small intersecting families; the basic problem is to decide how large they can be, and to describe the structure of the families that meet whatever bound we can derive.

The second version of the theorem is clearly more general than the first one, but has the weakness that its conclusion holds only if n is greater than some unspecified lower bound.

We call a collection of subsets of $\{1, \ldots, n\}$ a set system with underlying set $\{1, \ldots, n\}$; if the subsets all have size k we refer to it as a k-set system.

The easiest way to build a t-intersecting k-set system on an n-set is to simply take all k-subsets that contain a fixed t-set; clearly such a system has size

$$\binom{n-t}{k-t}.$$

We call a set system of this type a canonical t-intersecting family.

Lower bound for n

The lower bound on n in Theorem 2 is necessary because when n is not large enough, an intersecting family of maximal size need not be canonical.

Much work was devoted to determining the precise value of $f(n, k)$ needed.

Examples show that we need $n \geq (t+1)(k-t+1)$ for the bound to hold, and in 1978 Frankl proved that this constraint sufficed when t was large enough.

In 1984, Wilson proved that the bound in the EKR theorem holds if $n \geq (t+1)(k-t+1)$, and the characterisation holds provided $n > (t+1)(k-t+1)$.

In 1997 Ahlswede and Khachatrian determined the largest t-intersecting k-set systems on an n-set, for all values of n.

The result of this work is that, for each choice of n, k and t, we know that maximum size of the t-intersecting families and we know the structure of the families that reach this size.

The Hilton-Milner theorem (I)

Erdős, Ko and Rado conjectured that the largest 1-intersecting system that was not a subset of the canonical intersecting set system is the set of all k-subsets that contain at least two elements from a fixed set of three elements. It turns out that this conjecture is not true; the actual maximum sets were given by Hilton and Milner.

Hilton and Milner proved that the largest intersecting system that is not a subset of a canonical intersecting system can be constructed as follows. Let $\mathcal{F}_0$ be the set system of all k-sets that contain the element 1, and let $A = \{2, 3, \ldots, k+1\}$. Define $\mathcal{F}'$ to be the system of all the sets in $\mathcal{F}_0$ that intersect A, together with the set A. This system is intersecting and has size

$$\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1.$$

The Hilton-Milner theorem (II)

The next result is known as the Hilton–Milner theorem.

Theorem 3 (Hilton-Milner, 1967)

Let k and n be positive integers with $2 \leq k \leq \frac{n}{2}$. Let $\mathcal{A}$ be an intersecting k-set system on an n-set such that $\bigcap_{A \in \mathcal{A}} A = \emptyset$. Then

$$|\mathcal{A}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1.$$

Moreover, for $k > 3$, equality holds if and only if $\mathcal{A}$ is isomorphic to the system of all sets in $\mathcal{F}_0$ that intersect $\{2, \ldots, k+1\}$ together with the set $\{2, \ldots, k+1\}$; for $k = 3$ there are two non-isomorphic systems that meet this bound, the one described above and one more.

The goal of this lecture

In this minicourse, we consider the first way to extend the EKR theorem. In order to do this, we need to define the notion of 'intersecting' for the vertices of the graphs we consider. Once we have done this, we can define the notion of canonical intersecting families in these graphs.

In this lecture we consider two important families of strongly regular graphs for which the notion of canonical intersecting families (canonical cliques) is well defined:

- block graphs of orthogonal arrays (including Paley graphs of square order);
- block graphs of 2-$(n, m, 1)$ designs.

EKR properties

Given any graph X for which we can describe its canonical cliques (that is, typically cliques with large size and simple structure), we can ask whether X has any of the following three related Erdős-Ko-Rado (EKR) properties:

- EKR property: the clique number of X equals the size of canonical cliques.
- EKR-module property: the characteristic vector of each maximum clique in X is a $\mathbb{Q}$-linear combination of characteristic vectors of canonical cliques in X.
- strict-EKR property: each maximum clique in X is a canonical clique.

Given a graph X for which an analogue of the Erdős–Ko–Rado theorem is obtained, it is natural to asked whether an analogue of the Hilton-Milner theorem can be established (such a result is also called a stability result since this shows what are second largest maximal (w.r.t. inclusion) cliques in these graphs).

EKR-type results

The classical Erdős-Ko-Rado theorem [EKR61] classified maximum intersecting families of k-element subsets of $\{1, 2, \ldots, n\}$ when $n \geq 2k + 1$.

Since then, EKR-type results refer to understanding maximum intersecting families in a broader context, and more generally, classifying extremal configurations in other domains. The book [GM15] by Godsil and Meagher provides an excellent survey on the modern algebraic approaches to proving EKR-type results for permutations, set systems, orthogonal arrays, and so on.

[EKR61] P. Erdős, C. Ko, and R. Rado, Intersection theorems for systems of finite sets, Quart. J. Math. Oxford Ser. (2) 12 (1961), 313–320.

[GM15] C. D. Godsil, K. Meagher, *Erdős-Ko-Rado Theorems: Algebraic Approaches*, Cambridge University Press (2015).

Orthogonal arrays and their block graphs

An orthogonal array $OA(m,n)$ is an $m \times n^2$ array with entries from an n-element set T with the property that the columns of any $2 \times n^2$ subarray consist of all n^2 possible pairs.

The block graph of an orthogonal array $OA(m,n)$, denoted $X_{OA(m,n)}$, is defined to be the graph whose vertices are columns of the orthogonal array, where two columns are adjacent if there exists a row in which they have the same entry.

Let $S_{r,i}$ be the set of columns of $OA(m,n)$ that have the entry i in row r. These sets are cliques, and since each element of the n-element set T occurs exactly n times in each row, the size of $S_{r,i}$ is n for all i and r. These cliques are called the canonical cliques in the block graph $X_{OA(m,n)}$.

A simple combinatorial argument shows that the block graph of an orthogonal array is strongly regular. Moreover, by the Delsarte bound, a clique in $X_{OA(m,n)}$ has size at most n, and the canonical cliques show the tightness of this bound.

Intersecting columns in an arthogonal array

If we view columns of an orthogonal array that have the same entry in the same row as intersecting columns, then we can view the Delsate bound as the bound in the EKR theorem for intersecting columns of an orthogonal array. The question is, under what conditions will all cliques of size n in the graph $X_{OA}(m,n)$ be canonical? The following answer can be viewed as the uniqueness part of the EKR theorem.

Theorem 4 ([GM15, Corollary 5.5.3])

Let $X = X_{OA(m,n)}$ be the block graph of an orthogonal array $OA(m,n)$ with $n > (m-1)^2$. Then X has the strict-EKR property: the only maximum cliques in X are the columns that have entry i in row r for some $1 \leq i \leq n$ and $1 \leq r \leq m$.

This is equivalent to saying that the largest set of intersecting columns in an orthogonal array is the set of all columns that have the same entry in the some row, and these sets are the only maximum intersecting sets.

Open problems for the block graphs of orthogonal arrays

Problem 1

Find a characterisation of the orthogonal arrays, based only on the parameters of the array, for which all of the maximum cliques in the orthogonal array graph are canonical cliques.

Problem 2

Assume that $OA(m, (m-1)^2)$ is an orthogonal array and its orthogonal array graph has non-canonical cliques of size $(m-1)^2$. Do these non-canonical cliques form subarrays?

[GM15, Section 5.5] provides an example of non-canonical cliques in the block graph of an orthogonal array that form subarrays. In Lecture 4, we discuss a subfamily of affine polar graphs that can be viewed as block graphs of orthogonal arrays containing non-canonical cliques having subarray structure.

Problem 3

Determine all the maximum cliques in the block graph for any orthogonal array.

Peisert-type graphs

Let q be a prime power. Let $S \subset \mathbb{F}_{q^2}^*$ be a union of $m \leq q$ cosets of $\mathbb{F}_q^*$ in $\mathbb{F}_{q^2}^*$ such that $\mathbb{F}_q^* \subset S$, that is,

$$S = c_1\mathbb{F}_q^* \cup c_2\mathbb{F}_q^* \cup \cdots \cup c_m\mathbb{F}_q^*.$$

Then the Cayley graph $X = \mathrm{Cay}(\mathbb{F}_{q^2}^+, S)$ is said to be a Peisert-type graph of type (m, q). A clique in X is called a canonical clique if it is the image of the subfield $\mathbb{F}_q$ under an affine transformation.

A Peisert-type graph of type (m, q) can be viewed as a graph on the points of the affine geometry $AG(2, q)$ with two points being adjacent whenever the line through these points belongs to one of m prescribed parallel classes of lines. Peisert-type graphs are equivalent to the block graphs of orthogonal arrays obtained from the parallel classes of the affine plane $AG(2, q)$.

Another way to look at Peisert-type graphs is to consider them as fusions of certain amorphic cyclotomic association schemes.

Intersections of the class of Peisert-type graphs with some other classes

- Paley graphs $P(q^2)$ of square order are Peisert-type graphs;
- Peisert graphs $P^*(q^2)$, where $q \equiv 3 \pmod 4$, are Peisert-type graphs (not all Peisert graphs are Peisert-type graphs);
- Generalised Paley graphs $GP(q^2, d)$, where $d \mid (q+1)$ and $d > 1$ (not all generalised Paley graphs are Peisert-type graphs);
- Generalised Peisert graphs $GP^*(q^2, d)$, where $d \mid (q+1)$ and d is even (not all generalised Peisert graphs are Peisert-type graphs).

Balanced characteristic vectors

A vector in $\mathbb{R}^n$ is balanced if it is orthogonal to the all-ones vector $\mathbf{1}$. If v_S is the characteristic vector of a subset S of the set V, then we say that

$$v_S - \frac{|S|}{|V|}\mathbf{1}$$

is the balanced characteristic vector of S.

A question on Paley graphs of square order by Godsil and Meagher

Any clique of size q in $P(q^2)$ is a Delsarte clique, and so its balanced characteristic vector lies in the $\frac{-1+q}{2}$-eigenspace of the Paley graph. In [GM15, Section 5.9], a possible alternate proof of Blokhuis's result is presented; this proof relies on the following problems.

Problem 4

Show that the $\frac{-1+q}{2}$-eigenspace of $P(q^2)$ is spanned by the balanced characteristic vectors of the canonical cliques.

In [AGLY22], we showed that Problem 4 is equivalent to establishing the EKR-module property for Paley graphs of square order. Moreover, we established EKR-module property for all block graphs of orthogonal arrays (including Peisert-type graphs).

Problem 5

Prove that the only balanced characteristic vectors of sets of size q, in the $\frac{-1+q}{2}$-eigenspace of $P(q^2)$, are the balanced

Module method

In general, the module method (see [AM15, Section 4]) refers to the strategy of proving that a graph X satisfies the strict-EKR property in two steps:

- show that X satisfies the EKR-module property
- show that EKR-module property implies the strict-EKR property

As an example of the module method, [AM15, Theorem 4.5] provides a sufficient condition for the second step above for 2-transitive permutation groups.

[AM15] B. Ahmadi and K. Meagher, *The Erdős-Ko-Rado property for some 2-transitive groups*, Ann. Comb. 19 (2015), no. 4, 621–640.

Further, we list another recent results related to EKR-properties of Peisert-type graphs

Subspace structure of Delsarte cliques in Peisert-type graphs

The following theorem shows all maximum cliques in Peisert-type graphs of type (m, q), where $m \leq \frac{q+1}{2}$ have nice algebraic structure.

Theorem 5 ([AY22, Theorem 1.2])

Let X be a Peisert-type graph of type (m, q), where q is a power of an odd prime p and $m \leq \frac{q+1}{2}$. Then any maximum clique in X containing 0 is an $\mathbb{F}_p$-subspace of $\mathbb{F}_{q^2}$.

It is a good open problem to give symmetric constructions of Peisert-type graphs with $m > \frac{q+1}{2}$ with non-canonical cliques.

[AY22] S. Asgarli, C. H. Yip, *Van Lint-MacWilliams' conjecture and maximum cliques in Cayley graphs over finite fields*, J. Combin. Theory Ser. A **192** (2022), Paper No. 105667, 23 pp.
`https://doi.org/10.1016/j.jcta.2022.105667`

Erdős–Ko–Rado theorem in Peisert-type graphs (I)

The stability of canonical cliques in Peisert-type graphs has also been studied. For simplicity, we say a Peisert-type graph X with order q^2 has the property $ST(k)$ if each clique in X with size at least $q-k$ is contained in a canonical clique in X. From the definition, it is clear that the strict-EKR property is equivalent to the property $ST(0)$.

Theorem 6 ([Y23])

As $q \to \infty$, where q is a prime power, almost all Peisert-type graphs of type $(\frac{q+1}{2}, q)$ have the property $ST(\frac{\sqrt{q}}{2})$.

Theorem 7 ([Y23])

As $p \to \infty$, where p is prime, almost all Peisert-type graphs of type $(\lfloor \frac{2p-2}{3} \rfloor, p)$ have the strict-EKR property, and almost all Peisert-type graphs of type $(\frac{p+1}{2}, p)$ have the property $ST(p/20)$.

[Y23] C. H. Yip, *Erdős–Ko–Rado theorem in Peisert-type graphs*,
https://arxiv.org/abs/2302.00745

Erdős–Ko–Rado theorem in Peisert-type graphs (II)

Theorem 8 ([Y23])

Let $n \geq 2$ with the largest proper divisor being t and let $q = p^n$. As $p \to \infty$, almost all Peisert-type graphs of type $(q - o(p^t), q)$ do not have the strict-EKR property.

[Y23] C. H. Yip, *Erdős–Ko–Rado theorem in Peisert-type graphs*,
`https://arxiv.org/abs/2302.00745`

Further investigation

In Lecture 4, we will discuss a recent result stating that the non-canonical cliques (when exist) in the block graphs of orthogonal arrays with parameters $OA(\sqrt{q}+1, q)$ obtained from $AG(2, q)$ necessarily have the subarray structure.

An interesting project could arise from the investigation of the EKR properties of the block graphs of orthogonal arrays with parameters $OA(\sqrt{q}+1, q)$ obtained from affine planes different from $AG(2, q)$.

In particular, we plan to examine the database of small projective planes by Eric Moorhouse (the deletion of a line from projective plane together with its poits results in an affine plane that depends on the choice of the deleted line).

Designs

A 2-$(n, m, 1)$ design is a collection of m-sets of an n-set with the property that every pair from the n-set is in exactly one set.

A specific 2-$(n, m, 1)$ design is denoted by $(V, \mathcal{B})$, where V is the n-set (which we call the base set) and $\mathcal{B}$ is the collection of m-sets — these are called the blocks of the design.

A 2-$(n, m, 1)$ design may also be called a 2-design.

A simple counting argument shows that the number of blocks in a 2-$(n, m, 1)$ design is $\frac{n(n-1)}{m(m-1)}$ and each element of V occurs in exactly $\frac{n-1}{m-1}$ blocks (this is usually called the replication number).

EKR-type theorem for 2-designs

The blocks of a 2-design are a set system, and every pair from the base set occurs in exactly one block. Thus two distinct blocks of a 2-design must have intersection size 0 or 1. An intersecting set system from a 2-design is a set of blocks from the design in which any two have intersection of size exactly 1.

A question that naturally arises: what is the largest possible such set? Clearly if we take the collection of all blocks that contain a fixed element, we will have a system of size $\frac{n-1}{m-1}$.

An EKR-type theorem for 2-designs would state that this is the largest possible set of intersecting blocks and determine the conditions when the only intersecting sets of blocks that has this size is the set of all blocks that contain a fixed element. (The first result would be the bound in the EKR theorem, and the second would be the characterisation.)

Block graph of a 2-$(n, m, 1)$ design

The block graph of a 2-$(n, m, 1)$ design $(V, \mathcal{B})$ is the graph with the blocks of the design as the vertices in which two blocks are adjacent if and only if they intersect.

In a 2-design, any two blocks that intersect meet in exactly one point. The block graph of a design $(V, \mathcal{B})$ is denoted by $X_{(V,\mathcal{B})}$.

Alternatively, we could define a graph on the same vertex set in which two vertices are adjacent if and only if the blocks do not intersect — this graph is simply the complement of the block graph.

A clique in the block graph $X_{(V,\mathcal{B})}$ (or a coclique in its complement) is an intersecting set system from $(V, \mathcal{B})$.

Designs that are not symmetric are non-trivial

Fisher's inequality implies that the number of blocks in a 2-design is at least n; if equality holds, the design is said to be symmetric and the block graph of a symmetric 2-design is the complete graph K_n. To avoid this trivial case, we assume that our designs are not symmetric.

Theorem 9

The block graph of a 2-$(n, m, 1)$ design (that is not symmetric) is strongly regular with parameters

$$\left(\frac{n(n-1)}{m(m-1)}, \frac{m(n-m)}{m-1}, (m-1)^2 + \frac{n-1}{m-1} - 2, m^2\right).$$

Canonical cliques of the block graph of a non-trivial design

The Delsarte bound says that a clique in the block graph of a 2-$(n, m, 1)$ design has size at most $\frac{n-1}{m-1}$.

It is not difficult to construct a clique of this size: for any $i \in \{1, \ldots, n\}$ let S_i be the collection of all blocks in the design that contain i. We call the cliques S_i the canonical cliques of the block graph.

From this, we know that a set of intersecting blocks in a 2-design is no larger than the set of all blocks that contain a common point — this is the bound for an EKR-type theorem for the blocks in a design.

A sufficient condition for block graphs of 2-designs to have only canonical maximum cliques (I)

It is not known for which designs the canonical intersecting sets are the only maximum intersecting sets. Godsil & Meagher offer a partial result.

Theorem 10

If a clique in the block graph of a 2-$(n, m, 1)$ design does not consist of all the blocks that contain a given point, then its size is at most $m^2 - m + 1$.

Proof.

Assume that C is a non-canonical clique and that the set $\{1, \ldots, m\}$ is in C. Divide the other vertices in the clique into m groups, labeled G_i such that each vertex in group G_i contains the element i. Assume that G_1 is the largest group. Since the clique is non-canonical, there is a vertex in G_i for some $i > 1$.

A sufficient condition for block graphs of 2-designs to have only canonical maximum cliques (II)

All the vertices in G_1 must intersect this vertex so each vertex of G_1 must contain one of the $m-1$ elements in this vertex (but not the element i, as 1 and i are both in the set $\{1,\dots,m\}$). Since no two vertices of G_1 can contain the same element from the vertex in G_i , the size of G_1 can be no more than $(m-1)$. Since G_1 is the largest group the size of the clique is no more than $m(m-1)+1$. □

A corollary of this is an analogue of the EKR theorem, with the characterisation of maximum families, for intersecting sets of blocks in a 2-$(n,m,1)$ design.

Corollary 1 ([GM15, Corollary 5.3.5])

The only cliques of size $\frac{n-1}{m-1}$ in the block graph $X_{(V,\mathcal{B})}$ of a 2-$(n,m,1)$ design with $n > m^3-2m^2+2m$ are the sets of blocks that contain a given point i in $\{1,\dots,n\}$.

Example of the block graph of a 2-design for which there are non-canonical maximum clique

The characterisation in this corollary may fail if $n \leq m^3 - 2m^2 + 2m$.

For example, consider the projective geometry $PG(3,2)$. The points of this geometry can be identified with the 15 nonzero vectors in a 4-dimensional vector space V over $GF(2)$, and the lines with the 35 subspaces of dimension 2. This gives us a design with parameters 2-(15, 3, 1), where each block consists of the three nonzero vectors in a 2-dimensional subspace. There are exactly 15 subspaces of V with dimension 3, and each such subspace contains exactly seven points and exactly seven lines and so provides a copy of the projective plane of order two. In the block graph, the seven lines in any one of these projective planes forms a clique of size 7. In addition, each point of the design lies on exactly seven lines, and this provides a second family of 15 cliques of size 7.

Equality case

Theorem 11 ([GM15, Exercise 5.7])

In case $n = m^3 - 2m^2 + 2m$, *a non-canonical clique in the block graph of a 2-*$(n, m, 1)$ *design necessarily forms a* $(m^2 - m, m, 1)$ *subdesign (which is a projective plane of order* $m - 1$*).*

Open problems for block graphs of 2-$(n, m, 1)$ designs

Problem 6

Determine a characterisation of the 2-$(n, m, 1)$ designs, based only on the parameters of the design, for which the only maximum cliques in the block graph are the canonical cliques.

We have considered an example of a design with a block graph that has maximum cliques which are not canonical cliques. These maximum cliques have an interesting structure — namely, they form a subdesign isomorphic to the Fano plane. It is not clear if this a result of a wider phenomenon.

Problem 7

When the block graph of a design has maximum cliques that are not canonical, are the non-canonical cliques isomorphic to smaller designs?

Problem 8

Determine all the maximum cliques in the block graph for any 2-$(n, m, 1)$ design.

Further investigation

The stated problems suggest a comprehensive investigation of 2-$(n, m, 1)$ designs such that n and m do not satisfy the inequality.

Known infinite families of 2-designs

Let us have a look at the known infinite families of 2-desings (equivalently, Steiner systems $S(t, m, n)$ where $t = 2$). For this, let us have a look at item 5.11 in [Handbook of Combinatorial Designs, 2006, Edited By Charles J. Colbourn, Jeffrey H. Dinitz], which provides the following four known infinite families of Steiner systems $S(2, m, n)$:

1. $S(2, q+1, q^t + \ldots + q + 1)$, q a prime power, $t \geq 2$ (projective 2-designs);
2. $S(2, q, q^t)$, q a prime power, $t \geq 2$ (affine 2-designs);
3. $S(2, q+1, q^3+1)$, q a prime power;
4. $S(2, 2^r, 2^{r+s} + 2r - 2s)$, $2 \leq r < s$ (Denniston designs).

For family 3, the inequality on n and m is never satisfied. For family 5, if $s < 2r$ holds, the inequality is not satisfied. So both families 3 and 4 are non-trivial in the sense of EKR properties.

Projective 2-designs

Projective designs on points and lines in $PG(d, q)$:

- If $d \geq 4$, the inequality is satisfied, and we have only canonical cliques in the block graph.
- If $d = 3$, we have equality in this inequality (the inequality is not satisfied). The block graph in this case is the Grassmann graph $J_q(4, 2)$. The subgraph induced by the first neighbourhood of a given vertex is q-clique-extension of $(q+1) \times (q+1)$-lattice. A given vertex lies in $q+1$ canonical cliques (say, the rows of the extended lattice) and in $q+1$ non-canonical cliques (resp. the columns of the extended lattice).
- If $d = 2$, the inequality does not hold, but the design we have is given by the lines in a projective plane (the block graph is a clique and we have nothing to do).

Affine 2-designs

Affine designs on points and lines in $AG(d, q)$:

- If $d \geq 3$, the inequality is satisfied, and we have only canonical cliques in the block graph.
- If $d = 2$, the inequality does not hold, but the design we have is given by the lines in an affine plane (the block graph is a complete multipartite graph and all cliques are easy to describe).

Maximum cliques in the block graphs of projective and affine designs

Thus, maximum cliques in the block graphs of projective and affine designs on points and lines are known.

Concluding remarks

In this lecture we have discussed extensions of the EKR theorem to two important classes of strongly regular graphs (block graphs of orthogonal arrays (including Paley graphs of square order) and block graphs of 2-$(n, m, 1)$ designs). We have also formulated open problems and discussed possible directions for investigation.

In the next lecture we will discuss some conjectures whose statements can viewed as the Hilton-Milner theorem for Paley graphs of square order and special subclass of Peisert graphs.

Thank you for your attention!