Lecture 4: Extremal Peisert-type graphs without strict-EKR property

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What has been discussed so far?

- In Lecture 1, we discussed a proof that Paley graphs of square order have the strict-EKR property.
- In Lecture 2, we discussed EKR properties of graphs. In particular, we started a discussion on EKR properties of Peisert-type graphs, which are a class of graphs generalising Paley graphs of square order.
- In Lecture 3, we focused on possible analogues of Hilton-Milner theorem.

Some questions for graphs in flavour of EKR-type results

1. If we have a family of graphs, is it possible to define the notion of “intersecting” for vertices, define canonical cliques and show they are maximum?
2. If the answer for Question 1 is positive, is it possible to decide whether the canonical cliques are the only maximum cliques? If there are non-canonical cliques, can we enumerate them and describe their structure?
3. If the answer for Question 2 is positive (when the maximum cliques are known), is it possible to decide whether there exist maximal cliques that are not maximum. If there are any, can we enumerate second largest maximal cliques and describe their structure?

Peisert-type graphs

Given an abelian group G and a connection set $S \subset G \setminus \{0\}$ with $S = -S$, the Cayley graph $\mathrm{Cay}(G, S)$ is an undirected graph whose vertices are the elements of G, such that two vertices g and h are adjacent if and only if $g - h \in S$.

Let p be a prime and q a power of p. Let $\mathbb{F}_q$ be the finite field with q elements, $\mathbb{F}_q^+$ be its additive group, and $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$ be its multiplicative group.

Let $S \subset \mathbb{F}_{q^2}^*$ be a union of m cosets of $\mathbb{F}_q^*$ in $\mathbb{F}_{q^2}^*$, where $1 \leq m \leq q$, that is,

$$S = c_1\mathbb{F}_q^* \cup c_2\mathbb{F}_q^* \cup \cdots \cup c_m\mathbb{F}_q^*.$$

Then the Cayley graph $X = \mathrm{Cay}(\mathbb{F}_{q^2}^+, S)$ is said to be a Peisert-type graph of type (m, q).

Basic properties of Peisert-type graphs

In this lecture, we discuss many properties of Peisert-type graphs. Let us start with basic ones.

- The class of Paley graphs of square order is a special case of Peisert-type graphs.
- Every Peisert-type graphs of type (m, q) can be naturally defined on the points of the affine plane $\mathrm{AG}(2, q)$ with two points being adjacent whenver the line through these points belongs to one of m prescribed parallel classes of lines; the canonical cliques in a Peisert-type graph of type (m, q) are exactly the lines from m prescribed parallel classes defining the graph.
- The affine plane $\mathrm{AG}(2, q)$ can be viewd as an orthogonal $(q+1) \times q^2$-array $OA(q+1, q)$; every Peisert-type of type (m, q) graph can be viewed as the block graph of an orthogonal array $OA(m, q)$ obtained from this array $OA(q+1, q)$ by choosing the subset of m rows corresponding the m prescribed classes of parallel classes.

Number of pairwise non-isomorphic Peisert-type graphs

It is well-known that the stabiliser of the zero point of $\mathrm{AG}(2, q)$ acts 3-transitively on the set of lines through this point; this implies that, for any prime power q, all Peisert-type graphs of type $(3, q)$ are isomorphic.

Since the complement of a Peisert-type graph of type (m, q) is a Peisert-type graph of type $(q + 1 - m, q)$, we conclude that the number of pairwise non-isomorphic Peisert-type graphs of type (m, q) is equal to the number of pairwise non-isomorphic Peisert-type graphs of type $(q + 1 - m, q)$.

Whenever $q \leq 5$, there exists a unique Peisert-type graph of type (m, q) for any admissible value of m, that is, $1 \leq m \leq q$.

Extremal Peisert-type graphs without strict-EKR property

Theorem 1 ([AGLY22])

If $q > (m-1)^2$, then all Peisert-type graphs of type (m, q) have the strict-EKR property. Moreover, when q is a square, there exists a Peisert-type graph of type $(\sqrt{q}+1, q)$ without the strict-EKR property.

Given a prime power q, there exists the smallest value of m, say m_q, such that there exists a Peisert-type graph without strict-EKR property; we call such parameter m_q extremal.

In Theorem 1, we showed that if q is a square, then $m_q = \sqrt{q}+1$. In general, we have $m_q \geq \sqrt{q}+1$.

A Peisert-type graph of type (m_q, q) without strict-EKR property is called extremal.

[AGLY22] S. Asgarli, S. Goryainov, H. Lin, C. H. Yip, *The EKR-module property of pseudo-Paley graphs of square order*, Electron. J. Combin. **29** (2022), no. 4, #P4.33.

Number of pairwise non-isomorphic Peisert-type graphs

$q = 4:$

m	3	4
#Graphs	1	1
strict-EKR	-	-
without	1	1

$q = 5:$

m	3	4	5
#Graphs	1	1	1
strict-EKR	1	-	-
without	-	1	1

$q = 7:$

m	3	4	5	6
#Graphs	1	2	1	1
strict-EKR	1	2	-	-
without	-	-	1	1

$q = 8:$

m	3	4	5	6
#Graphs	1	1	1	1
strict-EKR	1	1	-	-
without	-	-	1	1

We thus have $m_4 = 3$, $m_5 = 4$, $m_7 = 5$ and $m_8 = 5$. In these four cases, an extremal graph is unique.

Number of pairwise non-isomorphic Peisert-type graphs

$$q = 9:$$

m	3	4	5	6	7
#Graphs	1	2	2	2	1
strict-EKR	1	1	1	-	-
without	-	1	1	2	1

$$q = 11:$$

m	3	4	5	6	7	8	9
#Graphs	1	2	2	4	2	2	1
strict-EKR	1	2	2	4	1	1	-
without	-	-	-	-	1	1	1

We thus have $m_9 = 4$, $m_{11} = 7$. In these two cases, an extremal graph is unique.

Number of pairwise non-isomorphic Peisert-type graphs

$q = 13:$

m	3	4	5	6	7	8	9	10	11
#Graphs	1	3	3	5	5	5	3	3	1
strict-EKR	1	3	3	5	5	4	2	-	-
without	-	-	-	-	-	1	1	3	1

$q = 16:$

m	3	4	5	6	7	8	9	10	11	12	13	14
#Graphs	1	2	3	4	5	6	6	5	4	3	2	1
strict-EKR	1	2	2	3	3	3	1	-	-	-	-	-
without	-	-	1	1	2	3	5	5	4	3	2	1

We thus have $m_{13} = 8$, $m_{16} = 5$. In these two cases, an extremal graph is unique.

Number of pairwise non-isomorphic Peisert-type graphs

$$q = 17 :$$

m	3	4	5	6	7	8	9	10	11	12	13	14	15
#Graphs	1	3	4	10	10	17	17	17	10	10	4	3	1
strict-EKR	1	3	4	10	10	17	17	16	9	5	1	-	-
without	-	-	-	-	-	-	-	1	1	5	3	3	1

We thus have $m_{17} = 10$, and an extremal graph is unique.

Number of pairwise non-isomorphic Peisert-type graphs

$$q = 19:$$

m	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
#Graphs	1	4	5	13	18	31	33	44	33	31	18	13	5	4	1
strict-EKR	1	4	5	13	18	31	33	44	32	30	14	5	-	-	-
without	-	-	-	-	-	-	-	-	1	1	4	8	5	4	1

We thus have $m_{19} = 11$, and an extremal graph is unique.

Number of pairwise non-isomorphic Peisert-type graphs

$$q = 23:$$

m	3	4	5	6	7	8	9	10	11	12	13
#Graphs	1	4	6	22	36	83	125	196	227	268	227
strict-EKR	1	4	6	22	36	83	125	196	227	268	226
without	-	-	-	-	-	-	-	-	-	-	1

m	14	15	16	17	18	19	20	21
#Graphs	196	125	83	36	22	6	4	1
strict-EKR	195	120	73	19	≥ 1	-	-	-
without	1	5	10	17	≥ 1	6	4	1

We thus have $m_{23} = 13$, and an extremal graph is unique.

Number of pairwise non-isomorphic Peisert-type graphs

$q = 25$:

m	3	4	5	6	7	8	9	10	11	12	13
#Graphs	1	4	7	19	34	79	132	223	293	379	391
strict-EKR	1	4	7	18	33	75	121	185	208	198	108
without	-	-	-	1	1	4	11	38	85	181	283

m	14	15	16	17	18	19	20	21	22	23
#Graphs	379	293	223	132	79	34	19	7	4	1
strict-EKR	34	3	1	-	-	-	-	-	-	-
without	345	290	222	132	79	34	19	7	4	1

We thus have $m_{25} = 6$, and an extremal graph is unique.

Main problem

This lecture is mainly devoted to the following problem.

Problem 1

Determine all extremal Peisert-type graphs without strict-EKR property.

Main results (I)

Theorem 2 ([GY23, Theorem 1.3])

Let $q = p^n$, where p is a prime and n is a positive integer. Let X be an extremal Peisert-type graph defined over $\mathbb{F}_{q^2}$.

- *If $n = 1$ and $p \geq 3$, then X is of type $(\frac{p+3}{2}, p)$ and X is unique up to isomorphism.*
- *If $n > 1$, then X is of type $(p^{n-k} + 1, q)$, where k is the largest proper divisor of n.*

Note that the result in the case $n = 1$ follows from some known results from the theory of directions in affine planes, and the case $n > 1$ was newly developed.

[GY23] S. Goryainov, C. H. Yip, *Extremal Peisert-type graphs without the strict-EKR property*, June 2023, `https://arxiv.org/abs/2206.15341`

Main results (II)

Theorem 3 ([GY23, Theorem 1.4])

Let q be the square of a prime power. There are exactly $(q+1)\sqrt{q}$ extremal Peisert-type graph defined over $\mathbb{F}_{q^2}$ and each Peisert-type graph of type $(3,q)$ is a subgraph of exactly one such extremal graph. Moreover, if X is such an extremal graph, then the following statements hold:

- *X is unique up to isomorphism: in fact, X is isomorphic to the affine polar graph $VO^+(4,\sqrt{q})$.*
- *X has exactly $\sqrt{q}+1$ canonical cliques containing 0, and $\sqrt{q}+1$ non-canonical cliques containing 0; moreover, these $2(\sqrt{q}+1)$ cliques lie in the same orbit under the action of the automorphism group of X.*
- *There is no Hilton-Milner type result: all maximal cliques in X are maximum cliques.*
- *The weight-distribution bound is tight for both non-principal eigenvalues of X.*

Main results (III)

Theorem 4 ([GY23, Theorem 1.5])

Let $q = r^3$, where r is a prime power and a non-square. There are exactly $r(r^5 + r^4 + r^3 + r^2 + r + 1)$ extremal Peisert-type graphs defined over $\mathbb{F}_{q^2}$. Moreover, if X is such an extremal graph, then the following statements hold:

- *X is unique up to isomorphism.*
- *Maximum cliques in X can be explicitly determined. In particular, X has exactly $r^2 + 1$ canonical cliques containing 0, and $r^2 + r + 1$ non-canonical cliques containing 0.*

Main results (IV)

Let $q = 2^5$. Let ε be a primitive element in $\mathbb{F}_q$ such that ε is a root of the irreducible polynomial $t^5 + t^2 + 1 \in \mathbb{F}_2[t]$. Let β be a root of the irreducible polynomial $t^2 + t + 1 \in \mathbb{F}_2[t]$; note that $\beta \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$. Consider the Peisert-type graph X_1 induced by the $\mathbb{F}_2$-subspace V_1 generated by the elements $\{1, \varepsilon, \beta, \varepsilon^{16}\beta, \varepsilon^{21} + \varepsilon^9\beta\}$, that is, $X_1 = \mathrm{Cay}(\mathbb{F}_{q^2}^+, V\mathbb{F}_q \setminus \{0\})$. Similarly, consider the Peisert-type graph X_2 induced by the $\mathbb{F}_2$-subspace V_2 generated by the elements $\{1, \varepsilon, \varepsilon^2, \varepsilon^3, \beta\}$.

For $i \in \{1, 2\}$, it is easy to verify that V_i is a non-canonical clique in X_i and X_i is a Peisert-type graph of type $(17, 32)$. Thus, in view of Theorem 2, X_1 and X_2 are extremal graphs. We have verified that X_1 is not isomorphic to X_2, showing that there are at least 2 non-isomorphic extremal Peisert-type graphs defined over $\mathbb{F}_{q^2}$.

Subspace structure of Delsarte cliques in Peisert-type graphs

Theorem 5 ([AY22, Theorem 1.2])

Let X be a Peisert-type graph of type (m, q), where q is a power of an odd prime p and $m \leq \frac{q+1}{2}$. Then any maximum clique in X containing 0 is an $\mathbb{F}_p$-subspace of $\mathbb{F}_{q^2}$.

[AY22] S. Asgarli, C. H. Yip, *Van Lint-MacWilliams' conjecture and maximum cliques in Cayley graphs over finite fields*, J. Combin. Theory Ser. A **192** (2022), Paper No. 105667, 23 pp.
https://doi.org/10.1016/j.jcta.2022.105667

An explicit infinite family of extremal Peisert-type graphs without strict-EKR property

Further, for any prime power q that is not a prime, we introduce an explicit infinite family of extremal Peisert-type graphs of type (m_q, q) without strict-EKR property and discuss its properties.

Graphs $Y_{q,n}$

Let $q = r^n$, where r is a prime power and n is prime. Assume $\mathbb{F}_{q^2} = \{x + y\beta : x, y \in \mathbb{F}_q\}$, where β is a root of an irreducible polynomial $f(t) = t^2 + dt + e \in \mathbb{F}_q[t]$.

Considering $\mathbb{F}_{r^n}$ as a n-dimensional $\mathbb{F}_r$-vector space underlying the affine space $\mathrm{AG}(n, r)$, let H be a an additive coset of a $(n-1)$-dimensional subspace in $\mathbb{F}_{r^n}$ (equivalently, let H be a hyperplane in $\mathrm{AG}(n, r)$). Note that $|H| = r^{n-1}$.

Let

$$S(H) = \mathbb{F}_q^* \cup \bigcup_{h \in H} (h + \beta)\mathbb{F}_q^*.$$

Let $Y_{q,n}(H)$ be the Peisert-type graph of type $(r^{n-1} + 1, r^n)$ defined by the generating set $S(H)$.

Proposition 1 ([GY23])

For any two hyperplanes H_1, H_2 in $AG(n, r)$, the graphs $Y_{q,n}(H_1)$ and $Y_{q,n}(H_2)$ are isomorphic.

We write $Y_{q,n}$ instead of $Y_{q,n}(H)$.

Given a prime power q, how many graphs $Y_{q,n}$ have we defined?

Let $q = p^m$, where p is prime and m is an integer, $m \geq 2$.

Let d be the number of different prime divisors of m. We have defined exactly d graphs of $Y_{q,n}$. Indeed, let $k_1, \ldots, k_d$ be the divisors of m such that

$$m/k_1, \ldots, m/k_d$$

are different primes and

$$m/k_1 < \ldots < m/k_d.$$

Put $n_i = m/k_i$ and $r_i = p^{k_i}$. Then, for any $i \in \{1, \ldots, d\}$, $q = r_i^{n_i}$ holds and we have defined the graphs $Y_{q,n_1}, \ldots, Y_{q,n_d}$.

Graphs $Y_{q,n}$ are Peisert-type graphs without strict-EKR property

Let $q = r^n$, where r is a prime power and n is a prime. Consider the graph $Y_{q,n}$.

Proposition 2 ([GY23])

The following statements hold.
(1) *The graph $Y_{q,n}$ is a Peisert-type graph of type $(r^{n-1}+1, r^n)$.*
(2) *The graph $Y_{q,n}$ fails to have the strict-EKR property.*

Conjecture 1

The graph $Y_{q,n}$ has exactly $(r^n-1)/(r-1)$ non-canonical cliques containing 0.

The conjecture was shown to be true when $n \in \{2, 3\}$.

A classification of extremal Peisert-type graphs without strict-EKR property

Theorem 6 ([GY23])

Let $q = p^m$, where p is prime and m is an integer $m \geq 2$. Let $k_1, \ldots, k_d$ be the divisors of m such that

$$m/k_1, \ldots, m/k_d$$

are different primes and

$$m/k_1 < \ldots < m/k_d.$$

Let $n_1 = m/k_1$ and $r_1 = p^{k_1}$. Then the following statements hold.
(1) Y_{q,n_1} is an extremal Peisert-type graph without strict-EKR property.
(2) If $n_1 \in \{2, 3\}$, then Y_{q,n_1} is the only (up to isomorphism) extremal Peisert-type graph without strict-EKR property.

Furher classification

If $q = 2^5$, then there exists at least two non-isomorphic extremal graphs without strict-EKR property ($Y_{32,5}$ and one more).

Problem 2

If $q = r_1^{n_1}$, r_1 and n_1 are as above, and $n_1 \geq 5$, how many pairwise non-isomorphic extremal Peisert-type graphs of type (m_q, q) without strict-EKR property does there exist?

Graphs $Y_{q,2}(\mathbb{F}_r)$ and X_q

Let $q = r^2$. Note that $\mathbb{F}_r$ is a hyperplane (a line) in $\mathrm{AG}(2, r)$. Consider the extremal Peisert-type graph $Y_{q,2}(\mathbb{F}_r)$ of type $(r+1, q)$. We have put $H = \mathbb{F}_r$ in the definition of $Y_{q,2}(H)$.

Let $Q = \{\gamma \in \mathbb{F}_q^* \mid \gamma^{r+1} = 1\}$.

Let $S = \bigcup_{\delta \in Q} (\delta + \beta)\mathbb{F}_q^*$.

Let X_q be the Peisert-type graph of type $(r+1, q)$ defined by the generating set S.

Theorem 7 ([GY23])

The graphs $Y_{q,2}(\mathbb{F}_r)$ and X_q are isomorphic.

Proof.

The generating set $S(\mathbb{F}_r)$ can be obtained from S by multiplication (from the left) by any non-degenerate matrix $\begin{pmatrix} \sigma & \sigma^r \\ 1 & 1 \end{pmatrix}$, where $\sigma \neq \sigma^r$. □

A non-canonical clique in X_q

Let ε be a primitive element of $\mathbb{F}_q$. Consider a 2-dimensional $\mathbb{F}_{\sqrt{q}}$-subspace in $\mathbb{F}_{q^2}$:

$$\begin{aligned} C_q &= (1+\beta)\mathbb{F}_{\sqrt{q}} + (\varepsilon^{\sqrt{q}} + \varepsilon\beta)\mathbb{F}_{\sqrt{q}} \\ &= \{(1+\beta)a + (\varepsilon^{\sqrt{q}} + \varepsilon\beta)b \mid a, b \in \mathbb{F}_{\sqrt{q}}\} \\ &= \{a + b\varepsilon^{\sqrt{q}} + (a + b\varepsilon)\beta \mid a, b \in \mathbb{F}_{\sqrt{q}}\} \\ &= \{(a + b\varepsilon)^{\sqrt{q}} + (a + b\varepsilon)\beta \mid a, b \in \mathbb{F}_{\sqrt{q}}\} \\ &= \{\gamma^{\sqrt{q}} + \gamma\beta \mid \gamma \in \mathbb{F}_q\} \\ &= \{\gamma(\gamma^{\sqrt{q}-1} + \beta) \mid \gamma \in \mathbb{F}_q\} \subset S \cup \{0\}. \end{aligned}$$

All non-canonical cliques in X_q

Proposition 3 ([GY23])

The following statements hold.

1. *The subspace C_q induces a non-canonical clique in X_q. Moreover, the intersection of any canonical clique in X_q containing* 0 *and C_q has exactly $\sqrt{q}-1$ nonzero elements (these elements are given by the elements $\gamma \in \mathbb{F}_q^*$ lying in the same coset of $\mathbb{F}_{\sqrt{q}}^*$ in $\mathbb{F}_q^*$.*
2. *For any $i \in \{0, 1, \ldots, \sqrt{q}\}$, the set $\varepsilon^i C_q$ induces a non-canonical clique in X_q, and, for any $i, j \in \{0, 1, \ldots, \sqrt{q}\}$ such that $i \neq j$, we have $\varepsilon^i C_q \cap \varepsilon^j C_q = \{0\}$.*
3. *The $\sqrt{q}+1$ non-canonical cliques $\{C_q, \varepsilon C_q, \varepsilon^2 C_q, \ldots, \varepsilon^{\sqrt{q}} C_q\}$ are the only non-canonical cliques in X_q containing 0.*

Hyperbolic quadric

Let V be a $(2e)$-dimensional vector space over a finite field $\mathbb{F}_q$, where $e \geq 2$ and q is a prime power, provided with the hyperbolic quadratic form

$$HQ(x) = x_1x_2 + x_3x_4 + \ldots + x_{2e-1}x_{2e}.$$

The set HQ^+ of zeroes of HQ is called the hyperbolic quadric, where e is the maximal dimension of a subspace in Q^+.

A generator of HQ^+ is a subspace of maximal dimension in HQ^+ known to be equal to e.

Affine polar graphs $VO^+(2e, q)$

Denote by $VO^+(2e, q)$ the graph on V with two vectors x, y being adjacent if and only if $HQ(x - y) = 0$. The graph $VO^+(2e, q)$ is known as an affine polar graph.

Lemma 1 ([BV22])

The graph $VO^+(2e, q)$ is a vertex-transitive strongly regular graph with parameters

$$\begin{aligned} v &= q^{2e} \\ k &= (q^{e-1}+1)(q^e-1) \\ \lambda &= q(q^{e-2}+1)(q^{e-1}-1)+q-2 \\ \mu &= q^{e-1}(q^{e-1}+1) \end{aligned} \tag{1}$$

and eigenvalues $r = q^e - q^{e-1} - 1$, $s = -q^{e-1} - 1$.

[BV22] A. E. Brouwer and H. Van Maldeghem, *Strongly Regular Graphs*, Cambridge University Press, Cambridge (2022).

X_q is isomoprhic to $VO^+(4, \sqrt{q})$

Let $V(n,r)$ be a n-dimensional vector space over the finite field $\mathbb{F}_r$, where $n \geq 2$ and r is a prime power. Let $f(x_1, x_2, \ldots, x_n) : V(n,r) \mapsto \mathbb{F}_r$ be a quadratic form on $V(n,r)$. Define a graph G_f on the set of vectors of $V(n,r)$ as follows:

$$\text{for any } u, v \in V(n,r), u \sim v \text{ if any only if } f(u-v) = 0.$$

Two quadratic forms $f_1(x_1, x_2, \ldots, x_n)$ and $f_2(y_1, y_2, \ldots, y_n)$ are said to be equivalent if there exists an invertible matrix $B \in GL(n,r)$ such that $f_1(Bx) = f_2(y)$.

Lemma 2

Let f_1 and f_2 be two equivalent quadratic forms. Then the graphs G_{f_1} and G_{f_2} are isomorphic.

Corollary 1 ([GY23])

The graphs X_q and $VO^+(4, \sqrt{q})$ are isomorphic.

Well-known facts about affine polar graphs $VO^+(2e, r)$

Proposition 4

The following statements hold.

1. *There is a one-to-one correspondence between generators of* HQ^+ *and maximal cliques in* $VO^+(2e, r)$ *containing the vector* 0.
2. *The graph* $VO^+(4, r)$ *has exactly* $2(r+1)$ *maximal cliques containing zero vector; these are the generators.*
3. *All maximal cliques of an affine polar graph* $VO^+(2e, r)$ *are equivalent under the action of the automorphism group.*
4. *An affine polar graph* $VO^+(2e, r)$ *is a rank 3 graph, that is, it is arc-transitive and its complement is arc-transitive.*

Thus, some of the properties of the extremal graph X_q are implications of known results on affine polar graphs.

Weight-distribution bound

The following lemma gives a lower bound for the number of non-zeroes (i.e., the cardinality of the support) for an eigenfunction of a strongly regular graph, known as the weight-distribution bound. It is a special case of a more general result for distance-regular graphs [KMP16, Section 2.4].

Lemma 3

Let X be a primitive strongly regular graph with non-principal eigenvalues θ_1 and θ_2, such that $\theta_2 < 0 < \theta_1$. Then an eigenfunction of X corresponding to the eigenvalue θ_1 has at least $2(\theta_1 + 1)$ non-zeroes, and an eigenfunction corresponding to the eigenvalue θ_2 has at least $-2\theta_2$ non-zeroes.

[KMP16] D. S. Krotov, I. Yu. Mogilnykh, V. N. Potapov, *To the theory of q-ary Steiner and other-type trades*, Discrete Mathematics 339 (3) (2016) 1150–1157.

Tightness of the weight-distribution bound for SRGs

The following lemma gives a combinatorial interpretation of the tightness of the weight-distribution bound in terms of special induced subgraphs.

Lemma 4

Let X be a primitive strongly regular graph with eigenvalues $\theta_2 < 0 < \theta_1$. Then the following statements hold.

(1) *For a θ_2-eigenfunction f, if the cardinality of support of f meets the weight-distribution bound, then there exists an induced complete bipartite subgraph in X with parts T_0 and T_1 of size $-\theta_2$. Moreover, up to multiplication by a constant, f has value* 1 *on the vertices of T_0 and value -1 on the vertices of T_1.*

(2) *For a θ_1-eigenfunction f, if the cardinality of support of f meets the weight-distribution bound, then there exists an induced pair of isolated cliques T_0 and T_1 in X of size $-\overline{\theta_2} = -(-1-\theta_1) = 1+\theta_1$. Moreover, up to multiplication by a constant, f has value 1 on the vertices of T_0 and value -1 on the vertices of T_1.*

Tightness of the weight-distribution bound for SRGs

(3) *If* X *has Delsarte cliques and each edge of* X *lies in a constant number of Delsarte cliques (for example,* X *is an edge-transitive strongly regular graph with Delsarte cliques), then any copy (as an induced subgraph) of the complete bipartite graph with parts of size* $-\theta_2$ *in* X *gives rise to an eigenfunction of* X *whose cardinality of support meets the weight-distribution bound and which is of the form given in item* (1)*.*

(4) *If the complement of* X *has Delsarte cliques and each edge of* X *lies in a constant number of Delsarte cliques (for example,* X *is a coedge-transitive strongly regular graph whose complement has Delsarte cliques), then any copy (as an induced subgraph) of a pair of isolated cliques of size* $\theta_1 + 1$ *in* X *gives rise to an eigenfunction of* X *whose cardinality of support meets the weight-distribution bound and which is of the form given in item* (2)*.*

An induced complete bipartite subgraph in X_q

Let $T_0 = Q$ and $T_1 = Q\beta$. Note that T_0 and T_1 are subsets of the lines with slopes 1 and β in $AG(2, q)$. These lines do not intersect with S and thus are cocliques in X_q, which means that T_0 and T_1 are cocliques.

Let $\gamma_1 \in T_0$ and $\gamma_2\beta \in T_1$ be two arbitrary elements from the cocliques T_0 and T_1. Consider their difference and take into account that Q is a subgroup of order $\sqrt{q} + 1$ in $\mathbb{F}_q^*$ and $-Q = Q$:

$$\gamma_2\beta - \gamma_1 = \gamma_2 + \gamma_1'\beta = \gamma_1'(\gamma_2(\gamma_1')^{-1} + \beta) = \gamma_1'(\gamma_2' + \beta) \in S,$$

where γ_1', γ_2' are uniquely determined elements from Q. This means that $T_0 \cup T_1$ induces a complete bipartite subgraph in X_q with parts T_0 and T_1 of size $\sqrt{q} + 1$.

WDB is tight for the negative eigenvalue of $X_q \simeq VO^+(4, \sqrt{q})$

Define a function $f : \mathbb{F}_{q^2} \mapsto \mathbb{R}$ by the following rule:

$$f(\gamma) = \begin{cases} 1, & \gamma \in T_0; \\ -1, & \gamma \in T_1; \\ 0, & \gamma \notin T_0 \cup T_1. \end{cases}$$

Proposition 5 ([GY23])

The function f is a $(-\sqrt{q} - 1)$-eigenfunction of X_q whose cardinality of support is $2(\sqrt{q} + 1)$.

Corollary 2 ([GY23])

The weight-distribution bound is tight for the negative non-principal eigenvalue $-\sqrt{q} - 1$ of $X_q \simeq VO^+(4, \sqrt{q})$.

Problem 3

Characterise $(-\sqrt{q} - 1)$-eigenfunctions of X_q whose cardinality of support meets the weight-distribution bound $2(\sqrt{q} + 1)$.

Orthogonal arrays and their block graphs

An orthogonal array $OA(m,n)$ is an $m \times n^2$ array with entries from an n-element set W with the property that the columns of any $2 \times n^2$ subarray consist of all n^2 possible pairs.

The block graph of an orthogonal array $OA(m,n)$, denoted $D_{OA(m,n)}$, is defined to be the graph whose vertices are columns of the orthogonal array, where two columns are adjacent if there exists a row in which they have the same entry.

Let $S_{r,i}$ be the set of columns of $OA(m,n)$ that have the entry i in row r. These sets are cliques, and since each element of the n-element set W occurs exactly n times in each row, the size of $S_{r,i}$ is n for all i and r. These cliques are called the canonical cliques in the block graph $D_{OA(m,n)}$.

A simple combinatorial argument shows that the block graph of an orthogonal array is strongly regular (see [GM15, Theorem 5.5.1]).

[GM15] C. D. Godsil, K. Meagher, *Erdös-Ko-Rado Theorems: Algebraic Approaches*, Cambridge University Press (2015).

Peisert-type graphs are block graphs of orthogonal arrays

In [AGLY22, Theorem 4], we explored the fact that each Peisert-type graph of type (m, q) can be realised as the block graph of an orthogonal array $OA(m, q)$. Moreover, there is a one-to-one correspondence between canonical cliques in the block graph and canonical cliques in a given Peisert-type graph.

In [GY23], we defined extremal Peisert-type graphs having non-canonical cliques. In fact, this definition can be naturally extended to the class of block graphs of orthogonal arrays obtained from affine planes different from $\mathrm{AG}(2, q)$ and having non-canonical cliques.

[AGLY22] S. Asgarli, S. Goryainov, H. Lin, C. H. Yip, *The EKR-module property of pseudo-Paley graphs of square order*, Electron. J. Combin. **29** (2022), no. 4, #P4.33.

[GY23] S. Goryainov, C. H. Yip, *Extremal Peisert-type graphs without the strict-EKR property*, June 2023, `https://arxiv.org/abs/2206.15341`

Complete orthogonal arrays

It is well known that an orthogonal array $OA(m, n)$ is equivalent to $m - 2$ mutually orthogonal Latin squares.

Further, it is well known that not more than $n - 1$ mutually orthogonal Latin squares of order n exist.

This implies that for an orthogonal array $OA(m, n)$, we necessarily have $m \leq n + 1$. A set of $n - 1$ mutually orthogonal Latin squares (an orthogonal array $OA(n + 1, n)$) is called complete.

It is well known that the existence of a complete set of mutually orthogonal Latin squares of order n (equivalently, a complete orthogonal array $OA(n + 1, n)$) is equivalent to the existence of a projective plane of order n, whose existence is known to be equivalent to the existence of an affine plane of order n.

Generalisation of the notion of extremality

Let A be a complete orthogonal array of type $OA(n+1, n)$ and let A_1 be an orthogonal array $OA(m, n)$ obtained as a subset of rows of A. For such an orthogonal array A_1 whose block graph has non-canonical cliques, we say that the block graph is A-extremal if the block graphs of all orthogonal arrays of type $OA(m-1, n)$ obtained as subsets of rows of A have the strict-EKR property.

In this sense, extremal Peisert-type graphs considered in this lecture are AG$(2, q)$-extremal.

A bound for block graphs of orthogonal arrays

Lemma 5 ([GM15, Corollary 5.5.3])

If $OA(m, n)$ is an orthogonal array with $n > (m-1)^2$, then the only cliques of size n in $D_{OA(m,n)}$ are canonical cliques.

Let $m-1$ be a prime power; then there exists an $OA(m, m-1)$ and, using MacNeish's construction [GM15, p. 98], it is possible to construct an $OA(m, (m-1)^2)$ from this array.

This larger orthogonal array has $OA(m, m-1)$ as a subarray, and thus the graph $D_{OA(m,(m-1)^2)}$ has the graph $D_{OA(m,m-1)}$ as an induced subgraph. Since this subgraph is isomorphic to $K_{(m-1)^2}$, it is a clique of size $(m-1)^2$ in $D_{OA(m,(m-1)^2}$ that is not canonical.

[GM15] C. D. Godsil, K. Meagher, *Erdös-Ko-Rado Theorems: Algebraic Approaches*, Cambridge University Press (2015).

Baer subplanes

Let Π be a finite projective (respectively, an affine) plane of order n and Π_0 a projective (respectively, an affine) subplane of Π of order n_0 different from Π; then $n_0 \leq \sqrt{n}$. If $n_0 = \sqrt{n}$, then Π_0 is called a Baer subplane of Π. Thus, Baer subplanes are the "biggest" possible proper subplanes of finite planes

Subarray structure of the non-canonical cliques in X_q

Problem 4 ([GM15, Problem 16.4.2])

Assume that $OA(m,(m-1)^2)$ *is an orthogonal array and its orthogonal array graph has non-canonical cliques of size* $(m-1)^2$*. Do these non-canonical cliques form subarrays that are isomorphic to orthogonal arrays with entries from* $\{1,\ldots,m-1\}$*?*

Proposition 6 ([GY23])

The non-canonical cliques in X_q *correspond to orthogonal subarrays* $OA(\sqrt{q}+1,\sqrt{q})$*, which are Baer subplanes in* $\mathrm{AG}(2,q)$*.*

The main implication of this result is that the block graphs of orthogonal arrays obtained from the affine planes $\mathrm{AG}(2,q)$ do not give a negative answer for Problem 4.

[GM15] C. D. Godsil, K. Meagher, *Erdös-Ko-Rado Theorems: Algebraic Approaches*, Cambridge University Press (2015).

Further problems

The following problem naturally arises.

Problem 5

Let A be an affine plane of order q such that q is the square of a prime power and A is not isomorphic to the affine plane $\mathrm{AG}(2, q)$. *Do the block graphs of the orthogonal arrays obtained from $\sqrt{q} + 1$ parallel classes of A have non-canonical cliques without a subarray (Baer-subplane) structure?*

In general, the following problem is of interest.

Problem 6

Let A be an affine plane of order q such that A is not isomorphic to an affine plane $\mathrm{AG}(2, q)$. *What are A-extremal block graphs of orthogonal arrays?*

Note that Problems 5 and 6 are special cases of the most general problem (see [GM15, Section 16.4]) of determination all the maximum cliques in the block graph of an orthogonal array.

Concluding remarks to the final lecture (I)

In this final lecture we have discussed extremal Peisert-type graphs of type (m_q, q) without strict-EKR property (that is, Peisert-type graphs having non-canonical cliques and the smallest possible number of canonical cliques). In particular, we determined the value m_q and explicitly constructed an extremal graph for every prime power q, an in case when q is prime, a square, or a cube but not a square, we showed the uniqueness of the extremal graph.

It is interesting question whether the uniqueness result extends to any other values of q.

Concluding remarks to the final lecture (II)

In a similar manner, given a prime power q, there exists the largest value of m, say M_q, such that there exists a Peisert-type graph of type (m, q) with strict-EKR property; one can also call such parameter M_q extremal.

A Peisert-type graph of type (M_q, q) with strict-EKR property is called extremal.

Problem 7

Given a prime power q, determine the value of M_q and characterise extremal Peisert-type graphs with strict-EKR property.

Concluding remarks to the minicourse

In the frame of this minicourse, we have formulated many conjectures and open problems. Personally, I am very interested in all of them and going to work on them with my colleagues and students. If you have interest in any of them, please let me know. I would be very happy to have new collaborators and thus more chances that these problems will be solved.

To conclude, I express my deep gratitude to the organisers and all participants.

Thank you for your attention!