Divisible design graphs from the symplectic graphs

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- New divisible designs from non-classical generalised quadrangles

Divisible design graphs

We consider simple graphs, that is graphs, without loops and multiple edges.

A k-regular graph on v vertices is called a divisible design graph with parameters $(v, k, \lambda_1, \lambda_2, m, n)$ if its vertex set can be partitioned into m classes of size n such that any two vertices from the same class has λ_1 common neighbours and any two vertices from different classes have λ_2 common neighbours.

In other words, a divisible design graph is a graph whose adjacency matrix is an incidence matrix of a (group) divisible design.

The partition of the vertex set of a divisible design graph into classes is called the canonical partition.

Motivation for the definition

In 1971, Rudvalis published [R71] the paper "(v, k, λ)-Graphs and polarities of (v, k, λ)-designs", where a one-to-one correspondence between (v, k, λ)-graphs and polarities, with no absolute points, of (v, k, λ)-designs was established. The motivation was that this gives an interplay between (strongly regular) graphs and (symmetric) designs. This connection can be useful for both parts. For example the easiest construction of a $(16, 6, 2)$ biplane is via the $(16, 6, 2)$-graph which is just the line graph of $K_{4,4}$.

In a similar way, divisible design graphs were first introduced in [HKM11] as a bridge between graph theory and theory of (group divisible) designs.

[HKM11] W. H. Haemers, H. Kharaghani, M. A. Meulenberg, *Divisible design graphs*, Journal of Combinatorial Theory, Series A, 118(3) (2011) 978–992.
`https://doi.org/10.1016/j.jcta.2010.10.003`

[R71] A. Rudvalis, *(v, k, λ)-Graphs and polarities of (v, k, λ)-designs*, Mathematische Zeitschrift, 120 (1971) 224–230.
`https://doi.org/10.1007/BF01117497`

Divisible design graphs: state of art

A complete bibliography on divisible design graphs can be found in [P]. In particular, a number of characterization results and explicit constructions are known.

Most of divisible design graphs with less than 40 vertices were enumerated in [PS22].

[P] D. Panasenko, Online repository of small strictly Deza graphs,
`http://alg.imm.uran.ru/dezagraphs/biblio.html`

[PS22] D. Panasenko, L. Shalaginov, *Classification of divisible design graphs with at most 39 vertices*, Journal of Combinatorial Designs, 30(4) (2022) 205–219.
`https://doi.org/10.1002/jcd.21818`

Overview of the new results (I)

Let q be an odd prime power.

We construct [DGHS24] a new family of divisible design graphs based on the symplectic graphs $Sp(4, q)$.

[DGHS24] B. De Bruyn, S. Goryainov, W. H. Haemers, L. Shalaginov, Divisible design graphs from the symplectic graph, April 2024.
`https://arxiv.org/abs/2404.09902`

Overview of the new results (II)

We also show that the complement of symplectic graphs $Sp(4, q)$ admits three kinds of equitable partitions, that satisfy the requirements of [HKM11, Construction 4.16] (this construction is known as the partial complement).

This gives rise to three more infinite families of divisible design graphs. The smallest graphs in these four families have 40 vertices and thus cannot be found in [PS22].

Note that at least two of the three kinds of equitable partitions extend to $Sp(2e, q)$, where $e \geq 2$.

[HKM11] W. H. Haemers, H. Kharaghani, M. A. Meulenberg, *Divisible design graphs*, Journal of Combinatorial Theory, Series A, 118(3) (2011) 978–992. `https://doi.org/10.1016/j.jcta.2010.10.003`

[PS22] D. Panasenko, L. Shalaginov, *Classification of divisible design graphs with at most 39 vertices*, Journal of Combinatorial Designs, 30(4) (2022) 205–219. `https://doi.org/10.1002/jcd.21818`

Overview of the new results (III)

The graph $Sp(4, q)$ can be viewed as the collinearity graph of the classical generalised quadrangle of type $\mathrm{GQ}(q, q)$, denoted by $W(3, q)$.

Let q be a power of 2. There is a non-classical construction of a generalised quadrangle of type $\mathrm{GQ}(q-1, q+1)$ based on a hyperoval embedded into $\mathrm{PG}(3, q)$. Recently, we were able to construct more infinite families of divisible design graphs based on these non-classical generalised quadrangles.

Strongly regular graphs

A k-regular graph on v vertices is called a strongly regular graph with parameters (v, k, λ, μ) if any two adjacent vertices have exactly λ common neighbours, and any two distinct non-adjacent vertices have exactly μ common neighbours.

If G is a strongly regular graph, then its complement is also a strongly regular graph. A strongly regular graph G is primitive if both G and its complement are connected. If G is not primitive, we call it imprimitive. The imprimitive strongly regular graphs are exactly the disjoint unions of complete graphs and their complements, namely, the complete multipartite graph.

We focus on primitive strongly regular graphs.

Projective space $\mathrm{PG}(d, q)$

Given an integer $d \geq 1$, a prime power q and a $(d+1)$-dimensional vector space W over $\mathbb{F}_q$, the projective space $\mathrm{PG}(d, q)$ of dimension d is an incidence system whose points, lines, ..., hyperplanes are, respectively, 1-dimensional, 2-dimensional, ..., d-dimensional subspaces in W, ordered by inclusion.

Note that the projective dimension is 1 less than the corresponding vector dimension.

Symplectic graph $Sp(2e, q)$ (I)

Let V be a $(2e)$-dimensional vector space over a finite field $\mathbb{F}_q$, where $e \geq 2$ and q is a prime power. For any nonzero $v \in V$, denote by $[v]$ the 1-dimensional subspace generated by v.

Let

$$K = \begin{pmatrix} 0 & I^{(e)} \\ -I^{(e)} & 0 \end{pmatrix}.$$

The symplectic graph $Sp(2e, q)$ relative to K over $\mathbb{F}_q$ is the graph with the set of 1-dimensional subspaces of V as its vertex set and the adjacency defined by

$[v] \sim [u]$ if and only if $vKu^t = 0$ for 1-dimensional subspaces $[v]$, $[u]$.

Equivalently, for arbitrary non-zero vectors $v = (v_1, \ldots, v_e, v_{e+1}, \ldots, v_{2e})$ and $u = (u_1, \ldots, u_e, u_{e+1}, \ldots, u_{2e})$, the vertices $[v]$ and $[u]$ are adjacent if and only if

$$\sum_{i=1}^{e} v_i u_{e+i} - \sum_{i=1}^{e} v_{e+i} u_i = 0.$$

Symplectic graph $Sp(2e, q)$ (II)

Lemma 1 ([BV22])

The graph $Sp(2e, q)$ is a rank 3 (in particular, arc-transitive) strongly regular graph with parameters

$$v = \frac{q^{2e} - 1}{q - 1}$$

$$k = \frac{q(q^{2e-2} - 1)}{q - 1}$$

$$\lambda = \frac{q^2(q^{2e-4} - 1)}{q - 1} + q - 1$$

$$\mu = \frac{k}{q} = \lambda + 2$$

and non-principal eigenvalues $r = q^{e-1} - 1$, $s = -q^{e-1} - 1$.

[BV22] A. E. Brouwer and H. Van Maldeghem, *Strongly Regular Graphs*, Cambridge University Press, Cambridge (2022).

Symplectic graph $Sp(4, q)$

The graph $Sp(4, q)$ can be viewed as a graph on the set of points of the projective space $\mathrm{PG}(3, q)$ with two points being adjacent whenever orthogonal.

The graph $Sp(4, q)$ is a strongly regular graph with parameters

$$\begin{aligned} v &= q^3 + q^2 + q + 1 = (q^2 + 1)(q + 1) \\ k &= q(q + 1) \\ \lambda &= q - 1 \\ \mu &= q + 1 \end{aligned}$$

and non-principal eigenvalues $r = q - 1$, $s = -q - 1$.

Delsarte-Hoffman bound

Let s be the smallest eigenvalue of a k-regular strongly regular graph G. Delsarte proved [D73] that the clique number of G is at most

$$1 - \frac{k}{s}.$$

This bound is known as the Delsarte-Hoffman bound (see [BCN89, Proposition 1.3.2]).

A clique in a strongly regular graph whose size attains the Delsarte-Hoffman bound is called a Delsarte clique (see [H21] for historical remarks).

[BCN89] A. E. Brouwer, A. M. Cohen, and A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag, Berlin (1989).

[D73] P. Delsarte. *An algebraic approach to the association schemes of coding theory*, Philips Res. Rep. Suppl., (10):vi+97, 1973.

[H21] W. H. Haemers, *Hoffman's ratio bound*, Linear Algebra and its Applications Volume 617, (2021) 215–219. `https://doi.org/10.1016/j.laa.2021.02.010`

Hoffman bound

Let s be the smallest eigenvalue of a k-regular graph G on v vertices. Hoffman proved that the independence number of G is at most

$$\frac{v}{1 - \frac{k}{s}}.$$

This bound is known as the Hoffman bound (see [BCN89, Proposition 1.3.2]).

A coclique in a regular graph whose size attains the Hoffman bound is called a Hoffman coclique.

A partition of the vertex set of a regular graph into Hoffman cocliques is called a Hoffman colouring.

Nexus of a Delsarte clique and a Hoffman coclique

Let G be a strongly regular graph with parameters (v, k, λ, μ) and smallest eigenvalue s and let C be a Delsarte clique in G. Then every vertex not in C has exactly $\frac{\mu}{-s}$ neighbours in C. The number $\frac{\mu}{-s}$ is called the nexus of the clique C.

Note that, if a strongly regular graph has Delsarte cliques, then all of them have the same nexus.

Let G be a regular graph with parameters and smallest eigenvalue s and let C be a Hoffman clique in G. Then every vertex not in C has exactly $-s$ neighbours in C. The number $-s$ is called the nexus of the coclique C.

Note that, if a regular graph has Hoffman cocliques, then all of them have the same nexus.

Delsarte cliques in symplectic graphs $Sp(2e, q)$

A totally isotropic subspace in $PG(2e-1, q)$ w.r.t. to a symplectic quadratic form is a subspace such that any two its points are orthogonal. It is known that maximal totally isotropic subspaces have projective dimension $e-1$ (vector dimension e) and size $(q^e-1)/(q-1)$.

It is known [BV22] that every maximal clique of $Sp(2e, q)$ is a Delsarte clique and that there is a one-to-one correspondence between Delsarte cliques in $Sp(2e, q)$ and maximal totally isotropic subspaces in PG$(2e-1, q)$.

Delsarte cliques in $Sp(4, q)$

For the graph $Sp(4, q)$, the maximal cliques are totally isotropic lines in $\mathrm{PG}(3, q)$ and have size $q + 1$; all these cliques are Delsarte cliques with nexus 1.

On the other hand, every line in $\mathrm{PG}(3, q)$ that is not totally isotropic necessarily forms a maximal coclique of size $q + 1$ in $Sp(4, q)$. Such a line is called hyperbolic.

Thus, there are two types of lines in $\mathrm{PG}(3, q)$ w.r.t. to a symplectic quadratic form: totally isotropic lines and hyperbolic lines, and they form all Delsarte cliques and all maximal cocliques of size $q + 1$ in $Sp(4, q)$, respectively.

Conjugate hyperbolic lines

The following lemma introduces a matching on the set of hyperbolic lines in $\mathrm{PG}(3,q)$.

Lemma 2

For any hyperbolic line ℓ in $PG(3,q)$ there exists a uniquely determined hyperbolic line ℓ' such that the points of $\ell \cup \ell'$ induce a complete bipartite graph $K_{q+1,q+1}$ with parts consisting of the points of ℓ and ℓ', respectively. Moreover, for any pair ℓ and ℓ' of such lines and for any point p not on ℓ and ℓ', the point p has exactly one neighbour in ℓ and exactly one neighbour in ℓ'.

Two hyperbolic lines in $\mathrm{PG}(3,q)$ that induce a complete bipartite graph $K_{q+1,q+1}$ are called conjugate.

Symplectic and special spreads

A spread in a three-dimensional projective space $\mathrm{PG}(3, q)$ is a set of lines such that each point of the space is incident with exactly one line.

A partition of the set of points of $\mathrm{PG}(2e-1, q)$ into maximal totally isotropic subspaces is called a symplectic spread. It is known [D77] that symplectic spreads exist for any integer $e \geq 2$ and prime power q. If $e = 2$, we get a symplectic spread consisting of totally isotropic lines in $\mathrm{PG}(3, q)$.

A spread S in $\mathrm{PG}(3, q)$, where q is odd, consisting of hyberbolic lines, having the property that, for any line $\ell \in S$, the conjugate line ℓ' also belongs to S, is called special.

Note that any two conjugate lines ℓ, ℓ' from a special spread S induce a complete bipartite graph $K_{q+1,q+1}$ in the symplectic graph $Sp(4, q)$; moreover, there exists a partition (associated with S) of the vertex set of $Sp(4, q)$ into such $K_{q+1,q+1}$'s.

[D77] R. H. Dye, *Partitions and their stabilizers for line complexes and quadrics*, Ann. Mat. Pura Appl. 114 (1977) 173–194

A special spread for $q = 3$

A regular graph is called a (0,2)-graph if any two distinct vertices have 0 or 2 common neighbours.

In [BO09], $(0, 2)$-graphs of valency 8 were classified. It turns out that the complement of one of the graphs is a divisible design graph with parameters $(40, 31, 22, 24, 10, 4)$. Moreover, the authors of [BO09] gave a construction for this graph based on a special spread for $q = 3$. We could generalize this construction.

[BO09] A. E. Brouwer, P. R. J. Östergård, *Classification of the* $(0, 2)$*-graphs of valency* 8, Discrete Mathematics, Volume 309, Issue 3, (2009) 532–547.
`https://doi.org/10.1016/j.disc.2008.07.037`

New divisible design graphs (I)

The core of our approach is the notion of a special spread. The following theorem constructively shows the existence of a special spread in $\mathrm{PG}(3, q)$ for any odd prime power q.

Theorem 1 ([DGHS24])

Given an odd prime power q, there exists at least one special spread in $\mathrm{PG}(3, q)$.

The following theorem thus gives a new construction for infinitely many divisible design graphs.

Theorem 2 ([DGHS24])

Consider $Sp(4, q)$ and a special spread S. Let $\Gamma_{q,S}$ be the complement of the graph obtained from $Sp(4, q)$ by removing all the edges of each $K_{q+1,q+1}$ in the partition of the vertex set $V(Sp(4, q))$ into $K_{q+1,q+1}$'s associated with the spread S. Then the graph $\Gamma_{q,S}$ is a divisible design graph with parameters

$$((q^2 + 1)(q + 1), q^3 + q + 1, q^3 - q^2 + q + 1, q^3 - q^2 + 2q, q^2 + 1, q + 1).$$

New divisible design graphs (II)

We have verified computationally that, for $q = 3, 5$ and 7, the projective space $\mathrm{PG}(3, q)$ has exactly $1, 2$ and 14 pairwise non-equivalent special spreads, respectively, and that non-equivalent special spreads give (in Theorem 2) non-isomorphic divisible design graphs. We generalise this phenomenon in the following theorem.

Theorem 3 ([DGHS24])

Let q be an odd prime power, and S_1 and S_2 be two non-equivalent special spreads in $\mathrm{PG}(3, q)$. Then the graphs Γ_{q,S_1} and Γ_{q,S_2} are not isomorphic.

Thus, the following purely geometric problem is of interest.

Problem 1

Given an odd prime power q, how many pairwise non-equivalent special spreads does there exist in $\mathrm{PG}(3, q)$? In other words, given an odd prime power q, how many pairwise non-isomorphic graphs does Theorem 2 produce?

Partial complements

Let X be a graph and $C_1, \ldots, C_t$ be a partition of the vertex set of X into t parts. Then the graph obtained from X by inverting adjacency between distinct parts and preserving the edges inside the parts is called the partial complement of X w.r.t. the partition $C_1, \ldots, C_t$.

Partial complements w.r.t. equitable partitions

Let G be a k-regular graph with the vertex set $V(G)$. Let $\Pi := (V_1, \ldots, V_t)$ be a partition of $V(G)$ into t parts (t-partition). The partition Π is said to be equitable t-partition if for any $i, j \in \{1, \ldots, t\}$ there is a constant p_{ij} such that any vertex from the part V_i is adjacent to precisely p_{ij} vertices from the part V_j.

The canonical partition of a DDG is always an equitable partition.

Lemma 3 ([HKM11, Construction 4.16], Partial complement)

Let Γ be a strongly regular graph with $\lambda = \mu$. If Γ has a Hoffman coloring, or an equitable partition into two parts of equal size, then the partial complement is a divisible design graph.

Note that the complement of $Sp(2e, q)$ is a strongly regular graph with the property $\lambda = \mu$. Further, we use special spreads one kind of equitable 2-partitions of $Sp(4, q)$, use symplectic spreads to construct two kinds of equitable 2-partitions of $Sp(2e, q)$ and apply Lemma 3 to produce three more infinite families of divisible design graphs.

New divisible design graphs (III)

The following theorem is an application of the partial complement to an equitable 2-partition based on a special spread.

Theorem 4 ([DGHS24])

Consider $Sp(4,q)$ and a special spread S. Partition the vertices of $Sp(4,q)$ into two parts V_1 and V_2 of equal size, such that, for every subgraph $K_{q+1,q+1}$ associated with S, one part is in V_1 and the other part is in V_2. Let $\Gamma'_{q,S}$ be the graph obtained from $Sp(4,q)$ by replacing the subgraphs induced by V_1 and V_2 by their complements. Then $\Gamma'_{q,S}$ is a divisible design graph with parameters

$$\left((q^2+1)(q+1), \frac{q^3+q^2+3q+1}{2}, \frac{q^3-q^2+3q+1}{2}, q^2+q, 2, \frac{(q^2+1)(q+1)}{2}\right).$$

An open problem

We verified that only one graph can be obtained by applying Theorem 4 to the special spread for $q = 3$.

Moreover, 12 and 16 graphs, respectively, can be obtained by applying Theorem 4 to the two special spreads for $q = 5$. Among these 28 graphs there are two pairs of isomorphic graphs. Thus, for $q = 5$, Theorem 4 gives 26 pairwise non-isomorphic graphs in total.

In case $q = 7$, we got at least 6000 graphs (from one spread) and stopped the search.

Problem 2

Given an odd prime power q, how many pairwise non-isomorphic graphs does Theorem 4 produce?

New divisible design graphs (IV)

A symplectic spread corresponds to a partition of the vertices of $Sp(2e,q)$ into Delsarte cliques. With a symplectic spread we can make two more families of divisible design graphs.

Theorem 5 ([DGHS24])

Consider $Sp(2e,q)$ with $e \geq 2$ and a symplectic spread R. Let $\Gamma_{q,e,R}$ be the graph obtained from $Sp(2e,q)$ by removing the edges of the cliques in the spread. Then $\Gamma_{q,e,R}$ is a divisible design graph with parameters

$$\left(\frac{q^{2e}-1}{q-1},\ q^e\frac{q^{e-1}}{q-1},\ q^e\frac{q^{e-2}-1}{q-1},\ \frac{(q^{e-1}-1)^2}{q-1},\ q^e+1,\ \frac{q^e-1}{q-1}\right).$$

In Theorem 5 we find just one divisible design graph for a given symplectic spread. This construction can be also interpreted as the partial complement with respect to a Hoffman colouring ([HKM11, Construction 4.16]).

New divisible design graphs (V)

Theorem 6 ([DGHS24])

Consider $Sp(2e, q)$ with $e \geq 2$ and q odd, and a symplectic spread R. Partition the vertices of $Sp(2e, q)$ into two parts V_1 and V_2 of equal size, such that each part contains $(q^e + 1)/2$ cliques of the spread. Let $\Gamma'_{q,e,R}$ be the graph obtained from $Sp(2e, q)$ by replacing the subgraphs induced by V_1 and V_2 by their complements. Then $\Gamma'_{q,e,R}$ is a divisible design graph with parameters

$$\left(v = \frac{q^{2e} - 1}{q - 1},\ \frac{v}{2} - q^{e-1},\ \frac{v}{2} - q^{2e-2} - q^{e-1},\ q^{2e-2} - q^{e-1},\ 2,\ \frac{v}{2}\right).$$

In Theorem 6 we obtain many non-isomorphic divisible design graphs with these parameters, because there are exponentially many choices for the partition into V_1 and V_2. This construction can be also interpreted as the partial complement with respect to an equitable 2-partition with equal sizes of parts ([HKM11, Construction 4.16]).

Open problems on symplectic spreads

In a similar way, we formulate the following open problems on symplectic spreads.

Problem 3

Given an odd prime power q, how many pairwise non-equivalent symplectic spreads does there exist in $\mathrm{PG}(3, q)$*?*

Problem 4

Given an odd prime power q, how many pairwise non-isomorphic graphs does Theorem 5 produce?

Problem 5

Given an odd prime power q, how many pairwise non-isomorphic graphs does Theorem 6 produce?

Generalised quadrangles

A generalised quadrangle is an incidence structure (P, B, I), with $I \subseteq P \times B$ an incidence relation, satisfying certain axioms. Elements of P are by definition the points of the generalized quadrangle, elements of B the lines. The axioms are the following:

1. There is an s ($s \geq 1$) such that on every line there are exactly $s + 1$ points. There is at most one point on two distinct lines.
2. There is a t ($t \geq 1$) such that through every point there are exactly $t + 1$ lines. There is at most one line through two distinct points.
3. For every point p not on a line L, there is a unique line M and a unique point q, such that p is on M, and q on M and L.

(s, t) are the parameters of the generalized quadrangle. The parameters are allowed to be infinite. If either s or t is 1, the generalised quadrangle is called trivial. A generalised quadrangle with parameters (s, t) is often denoted by $\mathrm{GQ}(s, t)$.

If (P, B, I) is a generalised quadrangle $\mathrm{GQ}(s, t)$, then (B, P, I^{-1}) is also a generalised quadrangle, of type $\mathrm{GQ}(t, s)$, called the dual GQ.

Collinearity graph of a generalised quadrangle

The collinearity graph of a generalised quadrangle is the graph having as vertices the points of the generalised quadrangle, with the collinear points connected.

It is well-known that the collinearity graph of a generalised quadrangle $\mathrm{GQ}(s,t)$ is a strongly regular graph with parameters

$$((s+1)(st+1), s(t+1), s-1, t+1).$$

Non-classical generalised quadrangles $T_2^*(O)$

Let O be a hyperoval in a (Desarguesian) projective plane π embedded into $\mathrm{PG}(3,q)$, where q is a power of 2.

Let $T_2^*(O)$ be the incidence structure whose points are the points of $\mathrm{PG}(3,q)$ that are not in π and whose lines are the lines of $\mathrm{PG}(3,q)$ that are not in π and pass through a point of O.

The incidence structure $T_2^*(O)$ is known to be a generalised quadrangle of type $\mathrm{GQ}(q-1,q+1)$.

Collinearity graph of $\mathrm{GQ}(q-1, q+1)$

The collinearity graph of a generalised quadrangle $\mathrm{GQ}(q-1, q+1)$ is a strongly regular graph with parameters

$$(q^3, (q-1)(q+2), q-2, q+2)$$

Then the complement has parameters

$$(q^3, (q+1)(q-1)^2, q^3-2q^2-q+4, q^3-2q^2-q+2),$$

and $\lambda = \mu + 2$, in particular.

A construction of divisible design graphs based on strongly regular graphs with $\lambda = \mu + 2$

The following construction of divisible design graphs was recently proposed by Panasenko & Shalaginov.

Theorem 7 ([PS22, Construction 16])

Let Γ be a strongly regular with $\lambda = \mu + 2$ admitting a Hoffman colouring $\{C_1, \ldots, C_m\}$(that is, a partition into Hoffman cocliques $C_1, \ldots, C_m$). Let Γ' be the graph obtained from Γ by joining each two vertices from C_i for every $i \in \{1, \ldots, m\}$. Then Γ' is a divisible design graph.

However, Panasenko & Shalaginov considered only finitely many examples of strongly regular graphs satisfying the conditions of this construction.

[PS22] D. Panasenko, L. Shalaginov, *Classification of divisible design graphs with at most 39 vertices*, Journal of Combinatorial Designs, 30(4) (2022) 205–219. `https://doi.org/10.1002/jcd.21818`

Examples from $T_2^*(O)$ (I)

Let p be a point of the hyperoval O. Then the sets of points of lines of $T_2^*(O)$ obtained from projective lines through p give a clique spread of the collinearity graph of $T_2^*(O)$, that is, give a Hoffman colouring in the complementary graph, which has $\lambda = \mu + 2$.

Thus, [PS22, Construction 16] produces infinitely many divisible design graphs.

[PS22] D. Panasenko, L. Shalaginov, *Classification of divisible design graphs with at most 39 vertices*, Journal of Combinatorial Designs, 30(4) (2022) 205–219. `https://doi.org/10.1002/jcd.21818`

Collinearity graph of $\mathrm{GQ}(q+1, q-1)$

The collinearity graph of a generalised quadrangle $\mathrm{GQ}(q+1, q-1)$ is a strongly regular graph with parameters

$$(q^2(q+2), q(q+1), q, q)$$

Examples from $T_2^*(O)$ (II)

The collinearity graph X_q of the dual of $T_2^*(O)$ is just the line graph of $T_2^*(O)$ with two lines being adjacent whenever they intersect.

Let $P_1, \dots, P_{q+2}$ be the points of the hyperoval O, and, for any i in $\{1, \dots, q+2\}$, let L_i be the set of lines of $T_2^*(O)$ obtained from the set of projective lines through P_i. Then for each i the set of lines L_i is a Hoffman coclique of size q^2 in X_q, and $L_1, \dots, L_{q+2}$ is a Hoffman colouring of X_q. The partial complement w.r.t. to this Hoffman colouring gives a divisible design graph.

Also, since q is even, it is possible to have an equitable 2-partition of X_q with parts of equal size where each part is a union of Hoffman cocliques. Then the partial complement w.r.t. to such an equitable 2-partition gives a divisible design graph.

Examples from $T_2^*(O)$ (III)

Let A be an affine plane in the affine space obtained from $\mathrm{PG}(3, q)$ by removing the projective plane π containing the hyperoval O, such that the line in π obtained by the projectivisation of A does not intersect the hyperoval O (a passant).

Let $Q_1, \ldots, Q_{q^2}$ be the points of A, and, for any j in $\{1, \ldots, q^2\}$, let M_j be the pencil of lines of $T_2^*(O)$ through Q_i.

Then for each j the set of lines L_j is a Delsarte clique of size $q+2$ in X_q, and $M_1, \ldots, M_{q^2}$ is a clique spread in X_q.

Since q is even, it is possible to have an equitable 2-partition of Y_q with parts of equal size where each part is a union of Delsarte cliques. Then the partial complement w.r.t. to such an equitable 2-partition gives a divisible design graph.

Thank you for your attention!