On maximal cliques in Paley graphs of square order

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based on joint works with

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Outline

- Paley graphs
- Maximum cliques in Paley graphs of square order (Erdös-Ko-Rado theorem for Paley graphs of square order)
- Second largest known maximal cliques in Paley graphs of square order (a conjecture on maximal cliques in Paley graphs of square order as an analogue of the Hilton-Milner theorem)
- The weight-distribution bound
- Generalised Paley graphs of square order
- Partial similar results on maximal cliques in generalised Paley graphs of square order

1. Paley graphs

Paley graph $P(q)$

We consider finite undirected graphs without loops and mulptiple edges.

Let q be an odd prime power, $q \equiv 1(4)$.

The Paley graph of order q (denoted by $P(q)$) is a graph defined as follows:

- the vertex set is the finite field $\mathbb{F}_q$;
- two vertices γ_1, γ_2 are adjacent iff $\gamma_1 - \gamma_2$ is a square in $\mathbb{F}_q^*$.

Since -1 is a square in $\mathbb{F}_q^*$ iff $q \equiv 1(4)$, the graph $P(q)$ is undirected.

Maximum and maximal cliques in $P(q)$

A clique (resp. coclique) is a set of pairwise adjacent (resp. non-adjacent) vertices.

Problem 1

What are maximum cliques (cocliques) in $P(q)$?

Since $P(q)$ is self-complementary, the studying cliques and the studying cocliques in $P(q)$ are equivalent.

Since $P(q)$ is strongly regular, we can apply Delsarte-Hoffman bound to $P(q)$. It says that a clique (coclique) in $P(q)$ has at most $\sqrt{q}$ vertices.

Problem 1 is unsolved in general.

2. Maximum cliques in Paley graphs of square order

Delsarte-Hoffman bound

For the clique number $\omega(\Gamma)$ of a distance-regular graph Γ (in particular, of a strongly regular graph), the Delsarte-Hoffman bound holds:

$$\omega(\Gamma) \leq 1 - \frac{k}{\theta_{\min}},$$

where $\theta_{\min}$ is the smallest eigenvalue of Γ.

A clique in a distance-regular graph whose size lies on the Delsarte-Hoffman bound is called a Delsarte clique.

The case of Paley graphs of square order q^2

Let q be an odd prime power.

According to the Delsarte-Hoffman bound, a clique in $P(q^2)$ has at most q vertices.

Since every element from $\mathbb{F}_q^*$ is a square in $\mathbb{F}_{q^2}^*$, the subfield $\mathbb{F}_q$ induces a clique of size q in $P(q^2)$, which implies the tightness of the Delsarte-Hoffman bound.

In 1984, Blokhuis classified maximum (Delsarte) cliques in $P(q^2)$ and proved [B84] that such a clique is an affine image of the subfield $\mathbb{F}_q$.

This result can be viewed as the analogue of Erdös-Ko-Rado theorem for Paley graphs of square order (see [GM15]).

[B84] A. Blokhuis, *On subsets of $GF(q^2)$ with square differences*, Indag. Math. **46** (1984) 369–372.

[GM15] C. D. Godsil, K. Meagher, *Erdös-Ko-Rado Theorems: Algebraic Approaches*, Cambridge University Press (2015).

EKR properties of Peisert-type graphs

Given any graph Γ for which we can describe its canonical cliques (that is, typically cliques with large size and simple structure), we can ask whether Γ has any of the following three related Erdős-Ko-Rado (EKR) properties:

- EKR property: the clique number of Γ equals the size of canonical cliques.
- EKR-module property: the characteristic vector of each maximum clique in Γ is a $\mathbb{Q}$-linear combination of characteristic vectors of canonical cliques in Γ.
- strict-EKR property: each maximum clique in Γ is a canonical clique.

EKR properties of Peisert-type graphs

EKR-properties of Peisert-type graphs (a family of Cayley graphs over finite fields that includes Paley graphs of square order) were independently studied in [AGLY22] and [LT22].

[AGLY22] S. Asgarli, S. Goryainov, H. Lin, C. H. Yip, *The EKR-module property of pseudo-Paley graphs of square order*, January 2022, `https://arxiv.org/abs/2201.03100`, accepted to the Electronic Journal of Combinatorics

[LT22] Cai Heng Li, Venkata Raghu Tej, On the EKR Module property. `https://arxiv.org/abs/2207.05947`

3. Second largest known maximal cliques in Paley graphs of square order

Second largest known maximal cliques in $P(q^2)$

Problem 2

What are maximal but not maximum cliques in $P(q^2)$?

Given an odd prime power q, put $r(q) := \begin{cases} 1, & q \equiv 1(4); \\ 3, & q \equiv 3(4). \end{cases}$

In 1996, Baker et al. found [2] maximal cliques of size $\frac{q+r(q)}{2}$ in $P(q^2)$ for any odd prime power q. Let us say that these cliques are of Type I.

In 2018, Goryainov et al. found [3] one more family of maximal cliques in $P(q^2)$ with the same size $\frac{q+r(q)}{2}$. Let us say that these cliques are of Type II.

[BEHW96] R. D. Baker, G. L. Ebert, J. Hemmeter, A. J. Woldar, *Maximal cliques in the Paley graph of square order*, J. Statist. Plann. Inference **56** (1996) 33–38.

[GKSV18] S. V. Goryainov, V. V. Kabanov, L. V. Shalaginov, A. A. Valyuzhenich, *On eigenfunctions and maximal cliques of Paley graphs of square order*, Finite Fields and Their Applications, **52**, (2018) 361–369.

Computations on maximal cliques of size $\frac{q+r(q)}{2}$ in $P(q^2)$

q	3	5	7	9	11	13	17	19	23
Clique size	3	3	5	5	7	7	9	11	13
#Orbits	1	1	1	3	3	4	9	4	4

q	25	27	29	31	37	41	43	47	49
Clique size	13	15	15	17	19	21	23	25	25
#Orbits	2	2	2	2	2	2	2	2	2

q	53	59	61	67	71	73	79	81	83
Clique size	27	31	31	35	37	37	41	41	43
#Orbits	2	2	2	2	2	2	2	2	2

Conjecture 1

For $q \geq 25$, the graph $P(q^2)$ contains exactly two non-equivalent cliques of size $\frac{q+r(q)}{2}$.

Finite field $\mathbb{F}_{q^2}$

Fix a non-square $d \in \mathbb{F}_q^*$.

Consider the polynomial $f(t) = t^2 - d \in \mathbb{F}_q[t]$.

Then

$$\mathbb{F}_{q^2} = \{x + y\alpha \mid x, y \in \mathbb{F}_q\},$$

where α is a root of $f(t)$.

Let β be a primitive element of $\mathbb{F}_{q^2}$.

Note that the elements from $\mathbb{F}_q^* = \langle \beta^{q+1} \rangle$ are squares in $\mathbb{F}_{q^2}^*$ because $q + 1$ is even.

Affine plane $\mathrm{AG}(2, q)$

Let $V(2, q)$ be a 2-dimensional vector space over $\mathbb{F}_q$.

Consider the affine plane $\mathrm{AG}(2, q)$ whose

- points are vectors of $V(2, q)$;
- lines are all cosets of 1-dimensional subspaces in $V(2, q)$;
- incidence relation is natural (whether a vector belongs to a coset).

Since $\mathbb{F}_{q^2}$ can be viewed as a 2-dimensional vector space over $\mathbb{F}_q$, the points of $\mathrm{AG}(2, q)$ can be matched with the elements of $\mathbb{F}_{q^2}$ as follows:

$$(x, y) \leftrightarrow x + y\alpha.$$

Quadratic and non-quadratic lines in $\mathrm{AG}(2, q)$

Given a line ℓ in $\mathrm{AG}(2, q)$, there exist elements $x_1 + y_1\alpha$ and $x_2 + y_2\alpha$ such that

$$\ell = \{x_1 + y_1\alpha + c(x_2 + y_2\alpha) \mid c \in \mathbb{F}_q\}.$$

The line ℓ is called quadratic (reps. non-quadratic) if $x_2 + y_2\alpha$ is a square (resp. non-square) in $\mathbb{F}_{q^2}^*$.

- The subfield $\mathbb{F}_q$ is a quadratic line.
- There are precisely $q + 1$ lines through a point: $\frac{q+1}{2}$ quadratic and $\frac{q+1}{2}$ non-quadratic lines.

$P(q^2)$ as a graph on points of the affine plane $\mathrm{AG}(2, q)$

For any distinct $\gamma_1, \gamma_2 \in \mathbb{F}_{q^2}$, the difference $\gamma_1 - \gamma_2$ is a square in $\mathbb{F}_{q^2}^*$ (equivalently, $\gamma_1 \sim \gamma_2$ in $P(q^2)$) iff the line connecting γ_1 and γ_2 is quadratic.

The automorphism group of $P(q^2)$

The automorphism group of $P(q^2)$ acts arc-transitively, and the following equality

$$\mathrm{Aut}\left(P(q^2)\right) = \left\{\, v \mapsto av^{\gamma} + b \mid a \in S,\ b \in \mathbb{F}_{q^2},\ \gamma \in \mathrm{Gal}(\mathbb{F}_{q^2}) \,\right\}$$

holds, where S is the set of square elements in $\mathbb{F}_{q^2}^*$.

The group $\mathrm{Aut}\left(P(q^2)\right)$ preserves the sets of quadratic and non-quadratic lines.

The group $\mathrm{Aut}\left(P(q^2)\right)$ has a subgroup that stabilises the quadratic line $\mathbb{F}_q$ and acts faithfully on the set of points that do not belong to $\mathbb{F}_q$; this subgroup is given by the affine transformations $x \mapsto ax + b$, where $a \in \mathbb{F}_q^*$ and $b \in \mathbb{F}_q$.

Geometric structure of maximal cliques of Type I

Take an element $\gamma \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$.

Since $\mathbb{F}_q$ is a quadratic line, the line through γ that is parallel to $\mathbb{F}_q$, is quadratic too.

The other $\frac{q-1}{2}$ quadratic lines through γ intersect $\mathbb{F}_q$ in $\frac{q-1}{2}$ points; denote this set of $\frac{q-1}{2}$ intersection points by X_γ.

For the conjugate element $\overline{\gamma}$, the equality $X_{\overline{\gamma}} = X_\gamma$ holds.

If $q \equiv 1(4)$, each of the sets $\{\gamma\} \cup X_\gamma$ and $\{\overline{\gamma}\} \cup X_\gamma$ induce a maximal clique of size $\frac{q+1}{2}$.

If $q \equiv 3(4)$, the set $\{\gamma, \overline{\gamma}\} \cup X_\gamma$ induces a maximal clique of size $\frac{q+3}{2}$.

[BEHW96] R. D. Baker, G. L. Ebert, J. Hemmeter, A. J. Woldar, *Maximal cliques in the Paley graph of square order*, J. Statist. Plann. Inference **56** (1996) 33–38.

The subgroup Q of order $q+1$ in $\mathbb{F}_{q^2}^*$

Put

$$\omega := \beta^{q-1}, Q := \langle \omega \rangle,$$

$$Q_0 := \langle \omega^2 \rangle, Q_1 := \omega \langle \omega^2 \rangle.$$

- Q is a subgroup of order $q+1$ in $\mathbb{F}_{q^2}^*$
- Q is the kernel of the norm mapping $N : \mathbb{F}_{q^2}^* \mapsto \mathbb{F}_q^*$; given an element $\gamma = x + y\alpha \in \mathbb{F}_{q^2}^*$,

$$N(\gamma) := \gamma^{q+1} = \gamma\gamma^q = \gamma\overline{\gamma} = x^2 - y^2 d$$

- Q forms an oval in $\mathrm{AG}(2,q)$ (that is a set of $q+1$ points with no three on a line)
- Q is included to the neighbourhood of 0
- If $q \equiv 1(4)$, then Q induces the complete bipartite graph with parts Q_0 and Q_1
- If $q \equiv 3(4)$, then Q induces a pair of disjoint cliques Q_0 and Q_1

Geometric structure of maximal cliques of Type II

If $q \equiv 1(4)$, each of the sets Q_0 and Q_1 induces a maximal coclique of size $\frac{q+1}{2}$ in $P(q^2)$ (a maximal clique of size $\frac{q+1}{2}$ in $\overline{P(q^2)}$).

If $q \equiv 3(4)$, each of the sets $\{0\} \cup Q_0$ and $\{0\} \cup Q_1$ induces a maximal clique of size $\frac{q+3}{2}$ in $P(q^2)$.

[GKSV18] S. V. Goryainov, V. V. Kabanov, L. V. Shalaginov, A. A. Valyuzhenich, *On eigenfunctions and maximal cliques of Paley graphs of square order*, Finite Fields and Their Applications, **52**, (2018) 361–369.

Consider the mapping $\varphi : \mathbb{F}_{q^2} \mapsto \mathbb{F}_{q^2}$ defined by the rule:

$$\varphi(\gamma) := \begin{cases} \frac{\gamma+1}{\gamma-1} & \text{if } \gamma \neq 1, \\ 1 & \text{if } \gamma = 1. \end{cases}$$

Proposition 1 ([GMS22])

For any $\gamma = x + y\alpha \in Q$, $\gamma \neq 1$, the equality $\varphi(\gamma) = \frac{y}{x-1}\alpha$ holds.

It means that φ maps $Q \setminus \{1\}$ to the line $\{c\alpha \mid c \in \mathbb{F}_q\}$.

Proposition 2 ([GMS22])

For any $\gamma = x + y\alpha \in Q$, $\gamma \neq 1$, the equality $\varphi(\gamma^2) = \frac{x}{yd}\alpha$ holds.

Theorem 1 ([GMS22])

If $q \equiv 1(4)$, then $\varphi(Q_0)$ is a maximal coclique of size $\frac{q+1}{2}$ and of Type I;
if $q \equiv 3(4)$, then $\varphi(Q_0 \cup \{0\})$ is a maximal clique of size $\frac{q+3}{2}$ and of Type I.

[GMS22] S. Goryainov, A. Masley, L. V. Shalaginov, *On a correspondence between maximal cliques in Paley graphs of square order*, Discrete Math. **345** (2022), no. 6, 112853.

More on the correspondence between maximal cliques in Paley graphs of square order

Let q_1 be an odd prime power, $q_1 \equiv 1(4)$. Let $\Gamma_{q_1}(0)$ and $\Gamma_{q_1}(1)$ denote the subgraphs of the Paley graph $P(q_1)$ induced by the set of nonzero squares S_{q_1} and and the set $1 + S_{q^2}$, respectively. Obviously, $\Gamma_{q_1}(0) \simeq \Gamma_{q_1}(1)$.

In [B00], for a Paley graph $P(q_1)$, it was considered the subgroup H in $\mathrm{Aut}(\Gamma_{q_1}(0))$ generated by the maps $\gamma \mapsto a\gamma$ where a is a nonzero square, the field automorphisms, and the map $\gamma \mapsto \gamma^{-1}$. In [MK05], it was shown that $\mathrm{Aut}(\Gamma_{q_1}(0)) = H$.

Let $q_1 = q^2$, where q is an odd prime power, and let $S_{q^2} := (\mathbb{F}_{q^2}^*)^2$.

[B00] A. E. Brouwer, *Locally Paley graphs*, Designs, Codes and Cryptography **21** (2000), 69–76.

[MK05] M. Muzychuk, I. Kovács, *A solution of a problem of A. E. Brouwer*, Designs, Codes and Cryptography **34**, 249–264 (2005).

More on the correspondence between maximal cliques in Paley graphs of square order

In [GMS22], the mapping

$$\varphi(\gamma) := \begin{cases} \frac{\gamma+1}{\gamma-1} & \text{if } \gamma \neq 1, \\ 1 & \text{if } \gamma = 1; \end{cases}.$$

was shown to be a correspondence between two known constructions of maximal cliques in $P(q^2)$ of Type II and Type I.

Proposition 3 ([B22])

The mapping φ restricted to $V(\Gamma_{q^2}(1))$ is an automorphism of $\Gamma_{q^2}(1)$.

[GMS22] S. Goryainov, A. Masley, L. V. Shalaginov, *On a correspondence between maximal cliques in Paley graphs of square order*, Discrete Math. **345** (2022), no. 6, 112853.

[B22] A. E. Brouwer, private communication.

More on the correspondence between maximal cliques in Paley graphs of square order

Consider the mapping φ' obtained from φ by shifting the preimage and image by 1. For this, consider the mapping $\gamma \mapsto \frac{\gamma+1}{\gamma-1}$ and change the variable by $\gamma = \gamma' + 1$.

Then we have $\varphi : \gamma' + 1 \mapsto \frac{\gamma'+2}{\gamma'}$ and $\varphi' : \gamma' \mapsto \frac{2}{\gamma'}$, for any $\gamma' \neq 0$.

Thus, the automorphism φ of $\Gamma_{q^2}(1)$ corresponds to the automorphism φ' of $\Gamma_{q^2}(0)$.

[B22] A. E. Brouwer, private communication.

A conjecture on maximal cliques in Paley graphs of square order

Conjecture 2

Each second largest maximal clique in $P(q^2)$, where $q \geq 25$, is equivalent to a clique of Type I or a clique of Type II.

In view of the correspondence φ, the cliques of Type I and of Type II (with a removed vertex) are equivalent as cliques in the local subgraph.

This means we can talk about a unique type of cliques in Conjecture 2.

Thus, a proof of Conjecture 2 would be an analogue of the well-known Hilton-Milner theorem on the largest intersecting families of r-element sets that are maximal but not maximum [Y22].

[Y22] C. H. Yip, private communication.

4. The weight-distribution bound

Eigenfunctions of graphs

Let $\Gamma = (V, E)$ be a k-regular graph on n vertices and λ be an eigenvalue of its adjacency matrix A. Let $u = (u_1, \ldots, u_n)^t$ be an eigenvector of A corresponding to λ. Then u defines a function $f_u : V \mapsto \mathbb{R}$, which is called a λ-eigenfunction of Γ.

For an eigenfunction f_u of Γ, the *support* is the set

$$Supp(f_u) := \{x \in V \mid f_u(x) \neq 0\}.$$

MS-problem

The following problem was first formulated in [VK15] (see also [SV21] for the motivation and details).

Problem 3 (MS-problem)

Given a graph Γ and its eigenvalue λ, find the minimum cardinality of the support of a λ-eigenfunction of Γ.

A λ-eigenfunction having the minimum cardinality of support is called optimal.

Problem 4 (Strong MS-problem)

Given a graph Γ and its eigenvalue λ, characterise optimal λ-eigenfunctions of Γ.

[VK15] K. V. Vorobev, D. S. Krotov, *Bounds for the size of a minimal 1-perfect bitrade in a Hamming graph*, Journal of Applied and Industrial Mathematics 9(1) (2015) 141–146.

[SV21] E. Sotnikova, A. Valyuzhenich, *Minimum supports of eigenfunctions of graphs: a survey*, Art Discrete Appl. Math. 4 (2021), no. 2, Paper No. 2.09, 34 pp.

A survey on Problem 4

Recently, Problem 4 was solved for several classes of graphs:

- all eigenvalues of Hamming graphs $H(n, q)$ when $q = 2$ or $q > 4$ and some eigenvalues of $H(n, q)$ when $q = 3, 4$;
- all eigenvalues of Johnson graphs (asymptotically);
- the smallest eigenvalue of Hamming, Johnson and Grassmann graphs;
- the largest non-principal eigenvalue of a Star graph S_n, $n \geq 8$;
- the largest non-principal eigenvalue of Doob graphs.

A survey on Problem 3

Excepting the results from the previous slide, Problem 3 was solved for several more classes of graphs:

- both non-principal eigenvalues of Paley graphs of square order;
- strongly regular bilinear forms graphs over a prime field.

Weight-distribution bound

Let Γ be a distance-regular graph of diameter $D(\Gamma)$ with intersection array $(b_0, b_1, \ldots, b_{D(\Gamma)-1}; c_1, c_2, \ldots, c_{D(\Gamma)})$ and nonprincipal eigenvalue λ.

Theorem 2 (Weight-distribution bound, [KMP16, Corollary 1])

A λ-eigenfunction f of Γ has at least $\sum_{i=0}^{D(G)} |W_i|$ nonzeros, where

$$W_0 = 1,$$

$$W_1 = \lambda$$

and

$$W_i = \frac{(\lambda - a_{i-1})W_{i-1} - b_{i-2}W_{i-2}}{c_i}.$$

[KMP16] D. S. Krotov, I. Yu. Mogilnykh, V. N. Potapov, *To the theory of q-ary Steiner and other-type trades*, Discrete Mathematics 339 (3) (2016) 1150–1157.

Known results when the weight-distribution bound is tight

- the eigenvalue -1 of the Boolean Hamming graph of an odd dimension and the minimum eigenvalue of an arbitrary Hamming graph;
- both non-principal eigenvalues of Paley graphs of square order;
- the minimum eigenvalue of Johnson graphs;
- the minimum eigenvalue of Grassmann graphs;
- the minimum eigenvalue of strongly regular bilinear forms graphs over a prime field.

Tightness of the weight-distribution bound for the smallest eigenvalue of a DRG

It was shown in [KMP16] that, for the smallest eigenvalue of a distance-regular graph Γ with a Delsarte clique, such that every edge is included in a constant number of Delsarte cliques, the tightness of the weight-distribution bound is equivalent to the existence of an isometric bipartite distance-regular induced subgraph $T_0 \cup T_1$, where T_0 and T_1 are parts, such that an optimal eigenfunction, up to multiplication by a non-zero constant, has the following form:

$$f(x) = \begin{cases} 1, & \text{if } x \in T_0; \\ -1, & \text{if } x \in T_1; \\ 0, & \text{otherwise.} \end{cases}$$

[KMP16] D. S. Krotov, I. Yu. Mogilnykh, V. N. Potapov, *To the theory of q-ary Steiner and other-type trades*, Discrete Mathematics 339 (3) (2016) 1150–1157.

The weight-distribution bound for a non-principal eigenvalue of an SRG

If Γ is a strongly regular graph with non-principal eigenvalues θ, τ, where $\tau < 0 < \theta$, then the following holds.

Theorem 3 ([KMP16], Weight-distribution bound for SRG))

(1) A τ-eigenfunction f of Γ has at least (-2τ) nonzeros;
(2) A θ-eigenfunction f of Γ has at least $2(\theta + 1)$ nonzeros.

[KMP16] D. S. Krotov, I. Yu. Mogilnykh, V. N. Potapov, *To the theory of q-ary Steiner and other-type trades*, Discrete Mathematics 339 (3) (2016) 1150–1157.

Tightness of the weight-distribution bound for the negative eigenvalue of a special SRG with a Delsarte clique

Let Γ be a strongly regular graph with a Delsarte clique such that every edge is included in a constant number of Delsarte cliques. Let τ be the negative eigenvalue of Γ and $\overline{\theta}$ be the positive non-principal eigenvalue of the complement $\overline{\Gamma}$. (Note that $\theta = -1 - \tau$ holds for all strongly regular graphs)

Let f be a τ-eigenfunction of Γ (a $\overline{\theta}$-eigenfunction of $\overline{\Gamma}$).

Theorem 4 (Follows from [KMP16])

(1) $|Supp(f)|$ *meets WDB if and only if there exists an induced complete bipartite subgraph with parts* T_0, T_1 *of size* $-\tau$;
(2) $|Supp(f)|$ *meets WDB if and only if there exists an induced disjoint union of two cliques* T_0, T_1 *of size* $\theta + 1$.

[KMP16] D. S. Krotov, I. Yu. Mogilnykh, V. N. Potapov, *To the theory of q-ary Steiner and other-type trades*, Discrete Mathematics 339 (3) (2016) 1150–1157.

Tightness of the weight-distribution bound for Paley graphs of square order

In [GKSV18], for Paley graphs $P(q^2)$, we showed the tightness of the weight-distribution bound for both non-principal eigenvalues, which are $\tau = \frac{-1-q}{2}$ and $\theta = \frac{-1+q}{2}$.

Let β be a primitive element in $\mathbb{F}_{q^2}$. Put $\omega := \beta^{q-1}$. Then $Q = \langle \omega \rangle$ is the subgroup of order $q+1$ in $\mathbb{F}_{q^2}^*$.

Facts about Q:

- Q is an oval in the corresponding affine plane;
- Q is the kernel of the norm mapping $N : \mathbb{F}_{q^2}^* \mapsto \mathbb{F}_q^*$, which means that $Q = \{\gamma \in \mathbb{F}_{q^2}^* \mid \gamma^{q+1} = 1\}$, or, equivalently, $Q = \{x + y\alpha \mid x, y \in \mathbb{F}_q, x^2 - y^2 d = 1\}$, where d is a non-square in $\mathbb{F}_q^*$ and $\alpha^2 = d$.

[GKSV18] S. Goryainov, V. Kabanov, L. Shalaginov, A. Valyuzhenich, *On eigenfunctions and maximal cliques of Paley graphs of square order*, Finite Fields and Their Applications 52 (2018) 361–369.

Tightness of the weight-distribution bound for Paley graphs of square order

Let $Q_0 = \langle \omega^2 \rangle$ and $Q_1 = \omega Q_0$.

Facts about Q:

- if $q \equiv 1(4)$, then $Q = Q_0 \cup Q_1$ induces a complete bipartite graph with parts Q_0 and Q_1;
- if $q \equiv 3(4)$, then $Q = Q_0 \cup Q_1$ induces a pair of disjoint cliques Q_0 and Q_1.

Corollary 1 ([GKSV18, Theorem 2])

The weight-distribution bound is tight for both non-principal eigenvalues of Paley graphs of square order.

[GKSV18] S. Goryainov, V. Kabanov, L. Shalaginov, A. Valyuzhenich, *On eigenfunctions and maximal cliques of Paley graphs of square order*, Finite Fields and Their Applications 52 (2018) 361–369.

5. Generalised Paley graphs of square order

Generalised Paley graphs of square order; WDB for the smallest eigenvalue

Let $m > 1$ be a positive integer. Let q be an odd prime power, $q \equiv 1 \ (2m)$. The m-Paley graph on $\mathbb{F}_q$, denoted $GP(q, m)$, is the Cayley graph $Cay(\mathbb{F}_q^+, (\mathbb{F}_q^*)^m)$, where $(\mathbb{F}_q^*)^m$ is the set of m-th powers in $\mathbb{F}_q^*$.

We consider the graphs $GP(q^2, m)$, where q is an odd prime power and m divides $q + 1$; these graphs are strongly regular and form a generalisation of Paley graphs of square order (the usual Paley graphs of square order are just 2-Paley graphs of square order).

The eigenvalues of $GP(q^2, m)$ are $\tau = (-\frac{q+1}{m})$ and $\theta = \frac{(m-1)q-1}{m}$.

Given an odd prime power q and an integer $m > 1$ such that m divides $q + 1$, a $(-\frac{q+1}{m})$-eigenfunction of the generalised Paley graph $GP(q^2, m)$ has at least $\frac{2(q+1)}{m}$ non-zeroes.

6. Partial similar results on maximal cliques in generalised Paley graphs of square order

Structure of Q (I)

Let us divide Q into m parts

$$Q = Q_0 \cup Q_1 \cup \ldots \cup Q_{m-1},$$

where $Q_0 = \langle \omega^m \rangle$, $Q_1 = \omega Q_0$, $\ldots$, $Q_{m-1} = \omega^{m-1} Q_0$.

Proposition 4 ([GSY22, Lemma 3.8])

Let q be a prime power and m be an integer such that $m > 1$, m divides $q + 1$. The mapping $\gamma \mapsto \gamma^{q-1}$ is a homomorphism from $\mathbb{F}_{q^2}^*$ to Q. Moreover, an element γ is an m-th power in $\mathbb{F}_{q^2}^*$ if and only if γ^{q-1} is an m-th power in Q.

[GSY22] S. Goryainov, L. Shalaginov, C. H. Yip, *On eigenfunctions and maximal cliques of generalised Paley graphs of square order*, arXiv:2203.16081

Structure of Q (II)

Let us divide Q into m parts

$$Q = Q_0 \cup Q_1 \cup \ldots \cup Q_{m-1},$$

where $Q_0 = \langle \omega^m \rangle$, $Q_1 = \omega Q_0$, $\ldots$, $Q_{m-1} = \omega^{m-1} Q_0$.

Proposition 5 ([GSY22, Lemma 3.10])

Let γ be an arbitrary element from Q, $\gamma \neq 1$. Then, for the image of $(\gamma - 1)$ under the action of the homomorphism, the following equality holds:

$$(\gamma - 1)^{q-1} = -\frac{1}{\gamma}.$$

[GSY22] S. Goryainov, L. Shalaginov, C. H. Yip, *On eigenfunctions and maximal cliques of generalised Paley graphs of square order*, arXiv:2203.16081

Structure of Q (III)

The following theorem basically states that each of the sets $Q_0, Q_1 \ldots, Q_{m-1}$ induces either a clique or an independent set, and there are at most two cliques among them.

Moreover, the theorem states that for every independent set Q_{i_1}, there exists uniquely determined independent set Q_{i_2} among $Q_0, Q_1 \ldots, Q_{m-1}$ such that there are all possible edges between Q_{i_1} and Q_{i_2} and there are no edges between Q_{i_1} and $Q \setminus Q_{i_2}$.

Structure of Q (IV)

Theorem 5 ([GSY22, Theorem 4.1])

Given an odd prime power q and an integer $m > 1$, m divides $q+1$, the following statements hold for the subgraph of $GP(q^2, m)$ induced by Q.
(1) If m divides $\frac{q+1}{2}$ and m is odd, then Q_0 is a clique, and $Q_1, \ldots, Q_{m-1}$ are independent sets; moreover, for any distinct i_1, i_2 such that $0 \le i_1 < i_2 \le m-1$, there are all possible edges between the sets Q_{i_1} and Q_{i_2} if $i_1 + i_2 \equiv 0 \pmod{m}$, and there are no such edges if $i_1 + i_2 \not\equiv 0 \pmod{m}$. In particular, $Q_1 \cup Q_{m-1}, \ldots, Q_{\frac{m-1}{2}} \cup Q_{\frac{m+1}{2}}$ induce $\frac{m-1}{2}$ complete bipartite graphs.

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Structure of Q (V)

(2) *If* m *divides* $\frac{q+1}{2}$ *and* m *is even, then* $Q_0, Q_{\frac{m}{2}}$ *are cliques, and* $Q_1, \ldots, Q_{\frac{m}{2}-1}, Q_{\frac{m}{2}+1}, \ldots, Q_{m-1}$ *are independent sets; moreover, for any distinct* i_1, i_2 *such that* $0 \leq i_1 < i_2 \leq m-1$*, there are all possible edges between the sets* Q_{i_1} *and* Q_{i_2} *if*

$$i_1 + i_2 \equiv 0 \ (mod\ m) \text{ and } \{i_1, i_2\} \neq \{0, \frac{m}{2}\}$$

and there are no such edges if

$$i_1 + i_2 \not\equiv 0 \ (mod\ m) \text{ or } \{i_1, i_2\} = \{0, \frac{m}{2}\}.$$

In particular, $Q_1 \cup Q_{m-1}, \ldots, Q_{\frac{m}{2}-1} \cup Q_{\frac{m}{2}+1}$ *induce* $(\frac{m}{2} - 1)$ *complete bipartite graphs.*

Structure of Q (VI)

(3) *If* m *does not divide* $\frac{q+1}{2}$, *then* m *is even.*
(3.1) *If* $\frac{m}{2}$ *is odd, then* $Q_0, Q_1, \ldots, Q_{m-1}$ *are independent sets; moreover, for any distinct* i_1, i_2 *such that* $0 \leq i_1 < i_2 \leq m-1$, *there are all possible edges between the sets* Q_{i_1} *and* Q_{i_2} *if*

$$i_1 + i_2 \equiv \frac{m}{2} \ (mod\ m),$$

and there are no such edges if

$$i_1 + i_2 \not\equiv \frac{m}{2} \ (mod\ m).$$

In particular, if $m = 2$, $Q = Q_0 \cup Q_1$ *is a complete bipartite graph; if* $m \geq 6$,
$Q_0 \cup Q_{\frac{m}{2}}, \ldots, Q_{\frac{m-2}{4}} \cup Q_{\frac{m+2}{4}}, Q_{\frac{m}{2}+1} \cup Q_{m-1}, \ldots, Q_{\frac{3m-2}{4}} \cup Q_{\frac{3m+2}{4}}$
induce $\frac{m}{2}$ *complete bipartite graphs.*

Structure of Q (VII)

(3.2) *If* $\frac{m}{2}$ *is even, then* $Q_{\frac{m}{4}}, Q_{\frac{3m}{4}}$ *are cliques, and* $Q_0, \ldots, Q_{\frac{m}{4}-1}, Q_{\frac{m}{4}+1}, \ldots, Q_{\frac{3m}{4}-1}, Q_{\frac{3m}{4}+1}, \ldots, Q_{m-1}$ *are independent sets; moreover, for any distinct* i_1, i_2 *such that* $0 \leq i_1 < i_2 \leq m-1$*, there are all possible edges between the sets* Q_{i_1} *and* Q_{i_2} *if*

$$i_1 + i_2 \equiv \frac{m}{2} \pmod{m} \text{ and } \{i_1, i_2\} \neq \{\frac{m}{2}, \frac{3m}{2}\},$$

and there are no such edges if

$$i_1 + i_2 \not\equiv \frac{m}{2} \pmod{m} \text{ or } \{i_1, i_2\} = \{\frac{m}{2}, \frac{3m}{2}\}.$$

In particular, if $m = 4$*,* $Q_0 \cup Q_2$ *is a complete bipartite graph; if* $m \geq 8$*, then* $Q_0 \cup Q_{\frac{m}{2}}, \ldots, Q_{\frac{m-4}{4}} \cup Q_{\frac{m+4}{4}}, Q_{\frac{m}{2}+1} \cup Q_{m-1}, \ldots, Q_{\frac{3m-4}{4}} \cup Q_{\frac{3m+4}{4}}$ *induce* $\frac{m-2}{2}$ *complete bipartite graphs.*

Structure of Q (VIII) and tightness of WDB for the smallest eigenvalue of $GP(q^2, m)$

Corollary 2 ([GSY22, Corollary 4.2])

Let q be an odd prime power and m be an integer $m \geq 2$, m divides $q+1$. Then, except for the case $m = 2$ and 2 divides $\frac{q+1}{2}$, there is at least one pair Q_{i_1}, Q_{i_2} among $Q_0, \ldots, Q_{m-1}$ such that $Q_{i_1} \cup Q_{i_2}$ induces a complete bipartite subgraph.

Corollary 3 ([GSY22, Theorem 4.3])

Let q be an odd prime power and m be an integer $m \geq 2$, m divides $q+1$. Then the weight-distribution bound is tight for the eigenvalue $(-\frac{q+1}{m})$ of $GP(q^2, m)$.

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Cliques of Type I and II for generalised Paley graphs of square order

Cliques of Type I and II can be similarly defined for generalised Paley graphs of square order.

Moreover, a similar correspondence φ between them can be established; this correspondence is also an automorphism of the subgraph induced by the neighbourhood of 1. This means that the cliques of Type I are maximal if and only if the cliques of Type II maximal.

Problem 5

Investigate the maximality or near-maximality of the cliques of Type I and Type II in generalised Paley graphs of square order.

Partial progress towards the solution of Problem 5 can be found in [GSY22].

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Thank you for your attention!