On two types of cliques related to eigenspaces of strongly regular graphs

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based on joint work with

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1. Equitable partitions of Latin-square graphs

Equitable partitions

A partition $\Delta = \{\Delta_1, \ldots, \Delta_r\}$ of the vertex set of a graph Γ is said to be equitable if there is an $r \times r$ matrix $M = (m_{ij})$ such that the number of vertices of Δ_j joined to a vertex $\omega \in \Delta_i$ is m_{ij}, depending on i and j but not on the choice of ω.

The spectrum of M is contained in the spectrum of the adjacency matrix $A(\Gamma)$ of the graph Γ since the characteristic polynomial of M divides that of $A(\Gamma)$ [GR01, Theorem 9.3.3].

The matrix M is called the quotient matrix of the equitable partition. When we speak of eigenvalues of an equitable partition, we refer to eigenvalues of the corresponding quotient matrix.

[GR01] C. Godsil and G. Royle, *Algebraic graph theory*, Graduate Texts in Mathematics, vol. 207, Springer-Verlag, New York, 2001.

Perfect sets

Equitable partitions are also called perfect colourings.

We use the term perfect for a set which is a part of a two-part equitable partition.

Perfect sets are also known as completely regular codes of radius 1.

General facts about equitable partitions (I)

Let Γ be a connected regular graph with valency k. Then k is a simple eigenvalue of $A(\Gamma)$.

Moreover, the quotient matrix M of an equitable partition has all row sums equal to k, so that k is an eigenvalue of M. We call k the principal eigenvalue.

We say that an equitable partition Δ is μ-equitable if its quotient matrix M has all nonprincipal eigenvalues equal to μ.

Furthermore, we call a nonempty proper subset S of Ω a μ-perfect set if the partition $\{S, \Omega \setminus S\}$ is μ-equitable. Note that, if a set S is μ-perfect, then so is its complement $\Omega \setminus S$.

General facts about equitable partitions (II)

Proposition 1.1

Let $\Delta = \{\Delta_1, \ldots, \Delta_r\}$ be a partition of the vertex set Ω of the regular connected graph Γ.
(a) If Δ is μ-equitable, then each set Δ_i is μ-perfect.
(b) Conversely, if $\Delta_1, \ldots, \Delta_{r-1}$ are all μ-perfect, then Δ is μ-equitable.

Corollary 1.2

Let S be a μ-perfect set, and T a nonempty proper subset of $\Omega \setminus S$. Then T is μ-perfect if and only if $S \cup T$ is μ-perfect.

Corollary 1.3

If Δ is a μ-equitable partition, then any nontrivial coarsening of Δ is μ-equitable.

Latin square graphs

A Latin square of order n is an $n \times n$ array with entries from an alphabet of n letters, such that each letter occurs once in each row and once in each column.

Given a Latin square L, we define the corresponding Latin square graph $\Gamma(L)$ whose vertices are the n^2 cells of the array L, two vertices are adjacent iff they lie in the same row or the same column or contain the same letter.

The graph $\Gamma(L)$ is strongly regular with eigenvalues

$$k = 3(n-1), \theta = n-3 \text{ and } \tau = -3$$

.

$(n-3)$-perfect sets given by a Delsarte clique

Proposition 1.4

Let Γ be a strongly regular graph with positive nonprincipal eigenvalue θ. Let S be a Delsarte clique in Γ. Then S is a θ-perfect set in Γ.

Proposition 1.5

Given a Latin square L of order n, a Delsarte clique in $\Gamma(L)$ necessary corresponds to cells lying in the same row, lying in the same column, or having the same letter.

Corollary 1.6

Let S be a row, a column, or a letter. Then S is a $(n-3)$-perfect set in $\Gamma(L)$.

$(n-3)$-perfect sets given by a corner

0	1	2	3	4
1	2	3	4	0
2	3	4	0	1
3	4	0	1	2
4	0	1	2	3

Proposition 1.7

Let L be the Cayley table of a cyclic group of order n and S be the left upper corner under the secondary diagonal. Then S is a $(n-3)$-perfect set in $\Gamma(L)$.

Inflation of Latin squares

Take a Latin square L_0 of order s.

Replace each occurrence of letter i by a Latin square of order t in alphabet A_i, where the alphabets for different letters are pairwise disjoint; this gives a Latin square L of order $n = st$.

Moreover, given an $(s-3)$-perfect set S_0 in L_0, the corresponding cells in L form an $(n-3)$-perfect set.

Main theorem

The following theorem exhaust (see [BCGG19]) the minimal $(n-3)$-perfect sets (that is, $(n-3)$-perfect sets which are not proper subsets of an $(n-3)$-perfect set).

Theorem 1.8 ([BCGG19, Theorem 5.5])

Let S be a minimal $(n-3)$-perfect set in the graph of a Latin square of order n. Then S is a row, a column, a letter, or an inflation of a corner set.

0	1	2
1	2	0
2	0	1

$\rightsquigarrow$

3	4	5	6	7	8
4	3	6	5	8	7
6	5	7	8	4	3
5	6	8	7	3	4
8	7	3	4	6	5
7	8	4	3	5	6

[BCGG19] R. A. Bailey, P. J. Cameron, A. L. Gavrilyuk and S. V. Goryainov, *Equitable partitions of Latin-square graphs*, Journal of Combinatorial Designs, Volume 17, Issue 3, March 2019, pages 142–160.

Main corollary

Corollary 1.9 ([BCGG19, Theorem 5.4])

Let $\Gamma(L)$ be the Latin-square graph defined by a Latin square of order n, and Δ a partition of the vertex set of $\Gamma(L)$. Then Δ is $(n-3)$-equitable if and only if each part of Δ is a disjoint union of rows, columns, letters, or inflations of corner sets.

Open problems (I)

Problem 1.1

Classify (-3)-perfect sets of Latin-square graphs.

Comment to Problem 1.1

Such a set S has the property that it meets any row, column or letter in a constant number s of cells, and its cardinality is sn. In particular, with $s = 1$, such set is a transversal. A classification of (-3)-perfect sets would imply a solution for the long standing Ryser's conjecture, asserting that any Latin square of odd order has a transversal. Many squares of even order do too, but some do not (for example, the Cayley table of the cyclic group). The conjecture is still open despite a lot of work, so characterising such sets is unlikely to be achieved soon.

Open problems (II)

Problem 1.2

Classify θ-perfect sets of the block-graphs of Steiner triple systems, where θ is the positive nonprincipal eigenvalue.

Comment to Problem 1.2

There are two known infinite families of strongly regular graphs with negative nonprincipal eigenvalue $\tau = -3$: Latin-square graphs and the block graphs of Steiner triple systems.

Problem 1.3

Some families of distance-regular graphs have nonempty intersection with the family of Latin-square graphs (for example, bilinear forms graph). Classify their equitable partitions.

Problem 1.4

Classify equitable partitions of graphs of mutually orthogonal Latin squares (that is, equitable partitions of the block graphs of orthogonal arrays).

2. Neumaier graphs

Definitions

A k-regular graph on v vertices is called edge-regular with parameters (v, k, λ) if every pair of adjacent vertices has λ common neighbours.

An edge-regular graph with parameters (v, k, λ) is called strongly regular with parameters (v, k, λ, μ) if every pair of distinct non-adjacent vertices has μ common neighbours.

A clique in a regular graph is called m-regular if every vertex that doesn't belong to the clique is adjacent to precisely m vertices from the clique. For an m-regular clique, the number m is called the nexus.

A question by Neumaier

For the clique number $\omega(\Gamma)$ of a strongly regular graph Γ, the Delsarte-Hoffman bound holds:

$$\omega(\Gamma) \leq 1 - \frac{k}{\tau},$$

where τ is the smallest eigenvalue of Γ.

A clique in a strongly regular graph is regular if and only if it has $1 - \frac{k}{\tau}$ vertices; such a clique is called a Delsarte clique.

In 1981, Neumaier proved [N81] that an edge-regular graph which is vertex-transitive, edge-transitive, and has a regular clique is strongly regular.

Neumaier then asked: "**Is it true that every edge-regular graph with a regular clique is strongly regular?**"

[N81] A. Neumaier, *Regular Cliques in graphs and Special $1\frac{1}{2}$-designs*, Finite Geometries and Designs, London Mathematical Society Lecture Note Series, 245–259 (1981).

Neumaier graphs

A non-complete edge-regular graph with parameters (v, k, λ) containing an m-regular s-clique is said to be a Neumaier graph with parameters $(v, k, \lambda; m, s)$.

A Neumaier graph that is not strongly regular is said to be a strictly Neumaier graph.

For a Neumaier graph, a spread is a partition of the vertex set into regular cliques.

Two constructions of strictly Neumaier graphs with 1-regular cliques

In [GK18], Greaves and Koolen constructed an infinite family of strictly Neumaier graphs with 1-regular cliques.

Gavrilyuk and Goryainov then searched for examples in a collection of known Cayley-Deza graphs [GS14] and found four more strictly Neumaier graphs with parameters $(24, 8, 2; 1, 4)$.

In [GK19], Greaves and Koolen found 'another' infinite family of strictly Neumaier graphs with 1-regular cliques, which contains one of the four graphs on 24 vertices.

[GK18] G. R. W. Greaves, J. H. Koolen, *Edge-regular graphs with regular cliques*, Europ. J. Combin., 71, 194–201 (2018).

[GK19] G. R. W. Greaves, J. H. Koolen, *Another construction of edge-regular graphs with regular cliques*, Discrete Mathematics, 342, Issue 10, (2019) 2818–2820.

[GS14] S. V. Goryainov, L. V. Shalaginov, *Cayley-Deza graphs with fewer than 60 vertices*, Siberian Electronic Mathematical Reports, 11, 268–310 (2014) (in Russian).

Strictly Neumaier graphs with 2^i-regular cliques

In [EGP19], Evans, Goryainov and Panasenko found a strictly Neumaier graph containing a 2^i-regular clique for every positive integer i.

The smallest graph in this family has parameters (16,9,4;2,4).

It was also proved that this graph on 16 vertices is the smallest strictly Neumaier graph (w.r.t the number of vertices).

[EGP19] R. J. Evans, S. V. Goryainov, D. I. Panasenko, The smallest strictly Neumaier graph and its generalisations, The Electronic Journal of Combinatorics, 26(2) (2019), #P2.29.

Affine polar graph

Let V be a $(2e)$-dimensional vector space over a finite field $\mathbb{F}_q$, where $e \geq 2$ and q is a prime power, provided with the hyperbolic quadratic form $Q(x) = x_1x_2 + x_3x_4 + \ldots + x_{2e-1}x_{2e}$.

The set Q^+ of zeroes of Q is called the hyperbolic quadric, where e is the maximal dimension of a subspace in Q^+. A generator of Q^+ is a subspace of maximal dimension e in Q^+.

Denote by $VO^+(2e, q)$ the graph on V with two vectors x, y being adjacent iff $Q(x - y) = 0$.

The graph $VO^+(2e, q)$ is known to be a vertex-transitive strongly regular graph with parameters

$$v = q^{2e}, k = (q^{e-1} + 1)(q^e - 1),$$

$$\lambda = q(q^{e-2} + 1)(q^{e-1} - 1) + q - 2, \mu = q^{e-1}(q^{e-1} + 1).$$

The Delsate cliques in $VO^+(2e, q)$ are necessarily cosets of the generators.

Affine polar graph

Note that $VO^+(2e, q)$ is isomorphic to the graph defined on the set of all $(2 \times e)$-matrices over $\mathbb{F}_q$

$$\{\begin{pmatrix} x_1 & x_3 & \dots & x_{2e-1} \\ x_2 & x_4 & \dots & x_{2e} \end{pmatrix}\},$$

where two matrices are adjacent iff the scalar product of the first and the second rows of their difference is equal to 0.

A spread in $VO^+(2e, q)$ is a set of q^e disjoint maximal cliques that correspond to all cosets of a generator.

It is known that the automorphism group of $VO^+(2e, q)$ acts transitively on the set of generators.

The smallest strictly Neumaier graph

Put $e = 2$ and $q = 2$, and consider the 1-dimensional subspace

$$W = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}.$$

The subspace W is contained in the two generators

$$W_1 = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix} \text{ and } W_2 = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}.$$

Take the vector

$$v = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and consider the cosets

$$v + W_1 = \begin{pmatrix} * & * \\ 1 & 0 \end{pmatrix}, \quad v + W_2 = \begin{pmatrix} * & 0 \\ 1 & * \end{pmatrix},$$

whose intersection is

$$v + W = \begin{pmatrix} * & 0 \\ 1 & 0 \end{pmatrix}.$$

The smallest strictly Neumaier graph

The cliques W_1 and $v + W_1$ lie in the same spread, have size 4 and are 2-regular, which means that the switching edges between these cliques preserves the regularity of the graph.

Moreover, the switching edges between the cliques W_1, $v + W_1$ gives a graph isomorphic to the complement of the Shrikhande graph.

The consequent switching edges between the cliques W_1, $v + W_1$ and then between the cliques W_2, $v + W_2$ gives the smallest strictly Neumaier graph, which is vertex-transitive, has parameters (16,9,4;2,4) and contains a spread.

A generalisation of the switching

This idea also works in the general case $e \geq 2$.

Take the $(e-1)$-dimensional subspace

$$W = \left(\begin{array}{ccc|cc} * & \dots & * & * & 0 \\ 0 & \dots & 0 & 0 & 0 \end{array}\right),$$

The subspace W is contained in the two generators

$$W_1 = \left(\begin{array}{ccc|cc} * & \dots & * & * & * \\ 0 & \dots & 0 & 0 & 0 \end{array}\right) \text{ and } W_2 = \left(\begin{array}{ccc|cc} * & \dots & * & * & 0 \\ 0 & \dots & 0 & 0 & * \end{array}\right).$$

Take the vector

$$v = \left(\begin{array}{ccc|cc} 0 & \dots & 0 & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 \end{array}\right)$$

and consider the cosets

$$v+W_1 = \left(\begin{array}{ccc|cc} * & \dots & * & * & * \\ 0 & \dots & 0 & 1 & 0 \end{array}\right), \quad v+W_2 = \left(\begin{array}{ccc|cc} * & \dots & * & * & 0 \\ 0 & \dots & 0 & 1 & * \end{array}\right),$$

whose intersection is

$$v + W = \left(\begin{array}{ccc|cc} * & \dots & * & * & 0 \\ 0 & \dots & 0 & 1 & 0 \end{array}\right).$$

A generalisation of the switching

The cliques W_1 and $v + W_1$ lie in the same spread, have size 2^e and are 2^{e-1}-regular, which means that the switching edges between these cliques preserves the regularity of the graph.

Moreover, the switching edges between the cliques W_1, $v + W_1$ gives a strongly regular graph which has parameters the same as the affine polar graph $VO^+(2e, 2)$.

Theorem 2.1 ([EGP19, Theorem 5.1])

The consequent switching edges between the cliques W_1, $v + W_1$ and then between the cliques W_2, $v + W_2$ gives a strictly Neumaier graph, which is not vertex-transitive and contains a 2^{e-1}-regular clique of size 2^e.

[EGP19] R. J. Evans, S. V. Goryainov, D. I. Panasenko, The smallest strictly Neumaier graph and its generalisations, The Electronic Journal of Combinatorics, 26(2) (2019), #P2.29.

Wang-Qiu-Hu switching

Let Γ be a graph whose vertex set is partitioned as $C_1 \cup C_2 \cup D$. Assume that $|C_1| = |C_2|$ and that the induced subgraphs on C_1, C_2, and $C_1 \cup C_2$ are regular, where the degrees in the induced subgraphs on C_1 and C_2 are the same. Suppose that all $x \in D$ satisfy one of the following

1. $|\Gamma(x) \cap C_1| = |\Gamma(x) \cap C_2|$, or
2. $\Gamma(x) \cap (C_1 \cup C_2) \in \{C_1, C_2\}$.

The graph Γ' obtained from Γ by modifying the edges between $C1 \cup C2$ and D as follows is cospectral with Γ:

$$\Gamma'(x) \cap (C_1 \cup C_2) := \begin{cases} C_1, & \text{if } \Gamma(x) \cap (C_1 \cup C_2) = C_2; \\ C_2, & \text{if } \Gamma(x) \cap (C_1 \cup C_2) = C_1; \\ \Gamma(x) \cap (C_1 \cup C_2), & \text{otherwise.} \end{cases}$$

[WQH19] W. Wang, L. Qiu, Y. Hu, Cospectral graphs, GM-switching and regular rational orthogonal matrices of level p, Linear Algebra and its Applications, Volume 563, 15 (2019), 154–177. [IM19] F. Ihringer, A. Munemasa, New strongly regular graphs from ginite geometries via switching, Linear Algebra and its Applications Volume 580, 1 November 2019, Pages 464–474.

Application of WQH-switching

Theorem 2.2 (Evans, G., 2022+)

The switching edges between two regular cliques of $VO^+(2e, 2)$ from the same spread can be viewed as WQH-switching.

Open problems (III)

Problem 2.1

Is it possible to construct new strictly Neumaier graphs from other classical polar spaces?

Problem 2.2

Is it possible to construct new strictly Neumaier graphs from other bent functions?

Problem 2.3

Is it possible to have a similar switching between regular cliques in other strongly regular graphs?

Comment to Problem 2.3

Together with Shamil Asgarli, Rhys Evans and Chi Hoi Yip, we have an ongoing project on switchings in Peisert-type graphs in characteristic 2.

3. Erdős-Ko-Rado (EKR) properties of Peisert-type graphs

Basic definitions

Let p be an odd prime, q a power of p. Let $\mathbb{F}_q$ be the finite field with q elements, $\mathbb{F}_q^+$ be its additive group, and $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$ be its multiplicative group.

Given an abelian group G and a connection set $S \subset G \setminus \{0\}$ with $S = -S$, the Cayley graph $\mathrm{Cay}(G, S)$ is the undirected graph whose vertices are elements of G, such that two vertices g and h are adjacent if and only if $g - h \in S$.

For a graph X, the clique number of X, denoted $\omega(X)$, is the size of a maximum clique of X.

EKR properties

Given any graph X for which we can describe its canonical cliques (that is, typically cliques with large size and simple structure), we can ask whether X has any of the following three related Erdős-Ko-Rado (EKR) properties:

- EKR property: the clique number of X equals the size of canonical cliques.
- EKR-module property: the characteristic vector of each maximum clique in X is a $\mathbb{Q}$-linear combination of characteristic vectors of canonical cliques in X.
- strict-EKR property: each maximum clique in X is a canonical clique.

EKR-type results

The classical Erdős-Ko-Rado theorem [EKR61] classified maximum intersecting families of k-element subsets of $\{1, 2, ..., n\}$ when $n \geq 2k + 1$.

Since then, EKR-type results refer to understanding maximum intersecting families in a broader context, and more generally, classifying extremal configurations in other domains. The book [GM15] by Godsil and Meagher provides an excellent survey on the modern algebraic approaches to proving EKR-type results for permutations, set systems, orthogonal arrays, and so on.

[EKR61] P. Erdős, C. Ko, and R. Rado, Intersection theorems for systems of finite sets, Quart. J. Math. Oxford Ser. (2) 12 (1961), 313–320.

[GM15] C. D. Godsil, K. Meagher, *Erdős-Ko-Rado Theorems: Algebraic Approaches*, Cambridge University Press (2015).

EKR-module property (I)

The EKR-type problems related to a transitive permutation group G can be reformulated in terms of the EKR properties of cocliques of the derangement graph $\Gamma(G)$, or equivalently, the cliques of the complement. Once we define canonical cocliques (or cliques), we can discuss the EKR properties of G after identifying G with $\Gamma(G)$.

The EKR-module property was first formally defined by Meagher [M19] in this context: a permutation group G naturally acts on the vector space W spanned by the characteristic vectors of canonical cliques, which makes W a G-module.

[M19] K. Meagher, *An Erdős-Ko-Rado theorem for the group* $PSU(3, q)$, Des. Codes Cryptogr. 87 (2019), no. 4, 717–744.

EKR-module property (II)

Each finite 2-transitive group has the EKR property [MSP16].

Meagher and Sin [MS21] recently showed that all finite 2-transitive groups have the EKR-module property. However, the strict-EKR property does not hold for permutations groups in general; recently, Meagher and Razafimahatratra [MR21] have shown that the general linear group $GL(2, q)$ is such a counterexample.

[MR21] K. Meagher and A. S. Razafimahatratra, *Erdős-Ko-Rado results for the general linear group, the special linear group and the affine general linear group*, arXiv:2110.08972

[MS21] K. Meagher and P. Sin, *All 2-transitive groups have the EKR-module property*, J. Combin. Theory Ser. A 177 (2021), Paper No. 105322, 21.

[MST16] K. Meagher, P. Spiga, and P. H. Tiep, *An Erdős-Ko-Rado theorem for finite 2-transitive groups*, European J. Combin. 55 (2016), 100–118.

EKR-module property (III)

We remark that our results are of similar flavour, although in our context of Peisert-type graphs, the corresponding vector space W does not carry a natural module structure. However, we remark that the definition of EKR-module property (even for permutation groups) does not need the additional G-module structure.

Module method

In general, the module method (see [AM15, Section 4]) refers to the strategy of proving that a graph Γ satisfies the strict-EKR property in two steps:

- show that Γ satisfies the EKR-module property
- show that EKR-module property implies the strict-EKR property

As an example of the module method, [AM15, Theorem 4.5] provides a sufficient condition for the second step above for 2-transitive permutation groups.

[AM15] B. Ahmadi and K. Meagher, *The Erdős-Ko-Rado property for some 2-transitive groups*, Ann. Comb. 19 (2015), no. 4, 621–640.

Blokhuis' result in terms of ERK properties

Consider the Paley graph P_{q^2} which is the Cayley graph defined on the additive group of $\mathbb{F}_{q^2}$, with the connection set being the set of squares in $\mathbb{F}^*_{q^2}$. Clearly, the subfield $\mathbb{F}_q$ forms a clique. Moreover, $a\mathbb{F}_q + b$ also forms a clique for each $a, b \in \mathbb{F}_{q^2}$ where a is a nonzero square. Such square translates of $\mathbb{F}_q$ are the canonical cliques [GM15, Section 5.9] in this example. Blokhuis proved that these are precisely the maximum cliques in P_{q^2}.

Theorem 3.1 ([B84, Theorem])

Let q be an odd prime power. The Paley graph P_{q^2} satisfies the strict-EKR property.

Godsil and Meagher [GM15, Section 5.9] call Theorem 3.1 the EKR theorem for Paley graphs.

[B84] A. Blokhuis, *On subsets of $GF(q^2)$ with square differences*, Indag. Math. **46** (1984) 369–372.

[GM15] C. D. Godsil, K. Meagher, *Erdős-Ko-Rado Theorems: Algebraic Approaches*, Cambridge University Press (2015).

Extensions and generalisations of Blokhuis' result

Extensions and generalisations of Theorem 3.1 can be found in [BF91],[S99],[M09],[AY21] and [AY21a]. A Fourier analytic approach was recently proposed in [Y21, Section 4.4].

[BF91] A. A. Bruen and J. C. Fisher, *The Jamison method in Galois geometries*, Des. Codes Cryptogr. 1 (1991), no. 3, 199–205.

[S99] P. Sziklai, *On subsets of* $GF(q^2)$ *with dth power differences*, Discrete Math. 208/209 (1999), 547–555.

[M09] N. Mullin, *Self-complementary arc-transitive graphs and their imposters* (2009). Master's thesis, University of Waterloo.

[AY21] S. Asgarli and C. H. Yip, *Rigidity of maximum cliques in pseudo-Paley graphs from unions of cyclotomic classes*, arXiv:2110.07176

[AY21a] S. Asgarli and C. H. Yip, *Van Lint-MacWilliams' conjecture and maximum cliques in Cayley graphs over finite fields*, arXiv:2106.01522

[Y21] C. H. Yip, *Gauss sums and the maximum cliques in generalised Paley graphs of square order*, Funct. Approx. Comment. Math. (2021)

Peisert-type graphs

While we have at least three different proofs of Theorem 3.1, all known proofs rely heavily on advanced tools such as the polynomial method over finite fields.

Instead, in this work, we follow a purely combinatorial approach. Although we are not able to give a simple proof of Theorem 3.1, we prove that a weaker version of Theorem 3.1 extends to a larger family of Cayley graphs, namely Peisert-type graphs.

Let q be an odd prime power. Let $S \subset \mathbb{F}_{q^2}^*$ be a union of $m \leq q$ cosets of $\mathbb{F}_q^*$ in $\mathbb{F}_{q^2}^*$ such that $\mathbb{F}_q^* \subset S$, that is,

$$S = c_1\mathbb{F}_q^* \cup c_2\mathbb{F}_q^* \cup \cdots \cup c_m\mathbb{F}_q^*.$$

Then the Cayley graph $X = \mathrm{Cay}(\mathbb{F}_{q^2}^+, S)$ is said to be a Peisert-type graph of type (m, q). A clique in X is called a canonical clique if it is the image of the subfield $\mathbb{F}_q$ under an affine transformation.

Some important examples of Peisert-type graphs

The following families of Cayley graphs are Peisert-type graphs (see [AY21a, Lemma 2.10]):

- Paley graphs of square order;
- Peisert graph with order q^2, where $q \equiv 3 \pmod 4$;
- Generalised Paley graphs $GP(q^2, d)$, where $d \mid (q+1)$ and $d > 1$;
- Generalised Peisert graphs $GP^*(q^2, d)$, where $d \mid (q+1)$ and d is even.

[AY21a] S. Asgarli and C. H. Yip, *Van Lint-MacWilliams' conjecture and maximum cliques in Cayley graphs over finite fields*, arXiv:2106.01522

Peisert-type graphs satisfy the EKR-module property

Blokhuis' theorem already implies that Paley graphs of square order possess the EKR-module property. In their book, Godsil and Meagher ask for an algebraic proof of this statement [GM15, Problem 16.5.1], which motivates our work.

Our main result answers this problem for a larger family of Cayley graphs:

Theorem 3.2 ([9, Theorem 1.3])

Peisert-type graphs satisfy the EKR-module property.

[AGLY22] S. Asgarli, S. Goryainov, H. Lin, C. H. Yip, *The EKR-module property of pseudo-Paley graphs of square order*, arXiv:2201.03100

[GM15] C. D. Godsil, K. Meagher, *Erdős-Ko-Rado Theorems: Algebraic Approaches*, Cambridge University Press (2015).

Orthogonal arrays and their block graphs

An orthogonal array $OA(m, n)$ is an $m \times n^2$ array with entries from an n-element set T with the property that the columns of any $2 \times n^2$ subarray consist of all n^2 possible pairs.

The block graph of an orthogonal array $OA(m, n)$, denoted $X_{OA(m,n)}$, is defined to be the graph whose vertices are columns of the orthogonal array, where two columns are adjacent if there exists a row in which they have the same entry.

Let $S_{r,i}$ be the set of columns of $OA(m, n)$ that have the entry i in row r. These sets are cliques, and since each element of the n-element set T occurs exactly n times in each row, the size of $S_{r,i}$ is n for all i and r. These cliques are called the canonical cliques in the block graph $X_{OA(m,n)}$. A simple combinatorial argument shows that the block graph of an orthogonal array is strongly regular, and, moreover, the canonical cliques are Delsarte cliques.

A sufficient condition for the block graph of an orthogonal array to have strict-EKR property

Theorem 3.3 ([GM16, Corollary 5.5.3], [AGLY22, Theorem 2.8])

Let $X = X_{OA(m,n)}$ be the block graph of an orthogonal array $OA(m,n)$ with $n > (m-1)^2$. Then X has the strict-EKR property: the only maximum cliques in X are the columns that have entry i in row r for some $1 \leq i \leq n$ and $1 \leq r \leq m$.

[AGLY22] S. Asgarli, S. Goryainov, H. Lin, C. H. Yip, *The EKR-module property of pseudo-Paley graphs of square order*, arXiv:2201.03100

[GM15] C. D. Godsil, K. Meagher, *Erdős-Ko-Rado Theorems: Algebraic Approaches*, Cambridge University Press (2015).

Connection between Peisert-type graphs and orthogonal arrays

The main ingredient in the proof of Theorem 3.2 is the following connection between Peisert-type graphs and orthogonal arrays, which is of independent interest.

Theorem 3.4 ([AGLY22, Theorem 1.4])

Each Peisert-type graph of type (m, q) can be realized as the block graph of an orthogonal array $OA(m, q)$. Moreover, there is a one-to-one correspondence between canonical cliques in the block graph and canonical cliques in a given Peisert-type graph.

We then were able to find two explicit eigenbases for the positive non-principal eigenvalue of the block graph of an orthogonal array, which led us to the result of Theorem 3.2 (more generally, it led us to the establishing of EKR-module property for the block graphs of orthogonal arrays).

[AGLY22] S. Asgarli, S. Goryainov, H. Lin, C. H. Yip, *The EKR-module property of pseudo-Paley graphs of square order*, arXiv:2201.03100

We remark that the idea of viewing certain Cayley graphs geometrically has appeared in the past; see for example [M09, Construction 5.2.1] and [AY21a, Section 4.2] for related discussion. However, Paley graphs and block graphs of orthogonal arrays are often treated independently; see for example [GM15, Chapter 5], and [AFMNSR21, Section 5]. Theorem 3.4 is the first to make an explicit connection between Peisert-type graphs and orthogonal arrays, and allows us to treat them in a uniform manner.

[M09] N. Mullin, *Self-complementary arc-transitive graphs and their imposters* (2009). Master's thesis, University of Waterloo.

[AY21a] S. Asgarli and C. H. Yip, *Van Lint-MacWilliams' conjecture and maximum cliques in Cayley graphs over finite fields*, arXiv:2106.01522

[GM15] C. D. Godsil, K. Meagher, *Erdős-Ko-Rado Theorems: Algebraic Approaches*, Cambridge University Press (2015).

[AFMNSR21] M. Adm, S. Fallat, K. Meagher, S. Nasserasr, M. N. Shirazi, and A. S. Razafimahatratra, Weakly Hadamard diagonalizable graphs, Linear Algebra Appl. 610 (2021), 86–119

Strongly regular graphs due to Brouwer, Wilson, and Xiang that generalise Peisert-type graphs

It is known that the block graph of an orthogonal array is strongly regular. Thus, Theorem 3.4 also implies the same conclusion for the Peisert-type graphs. We remark that Peisert-type graphs form a subfamily of a well-known family of strongly regular Cayley graphs defined on finite fields due to Brouwer, Wilson, and Xiang [12]: the connection set is a union of semi-primitive cyclotomic classes of $\mathbb{F}_{q^2}$. However, their proof heavily relied on the fact we can compute semi-primitive Gauss sums explicitly using Stickelberger's theorem and its variants; see [BWX99, Proposition 1] and [AY21, Corollary 3.6]. Theorem 3.4 can be proved using a purely combinatorial argument, thus giving an elementary proof of the corollary below.

[AY21] S. Asgarli and C. H. Yip, *Rigidity of maximum cliques in pseudo-Paley graphs from unions of cyclotomic classes*, arXiv:2110.07176

[BWX99] A. E. Brouwer, R. M. Wilson, and Q. Xiang, *Cyclotomy and strongly regular graphs*, J. Algebraic Combin. 10 (1999), no. 1, 25–28.

Peisert-type graphs are strongly regular

Corollary 3.5 ([AGLY22, Corollary 1.5])

A Peisert-type graph of type (m, q) is strongly regular with parameters $(q^2, m(q-1), (m-1)(m-2)+q-2, m(m-1))$ and eigenvalues $k = m(q-1)$ (with multiplicity 1), $-m$ (with multiplicity q^2-1-k) and $q-m$ (with multiplicity k). In particular, a Peisert-type graph of type $(\frac{q+1}{2}, q)$ is a pseudo-Paley graph.

[AGLY22] S. Asgarli, S. Goryainov, H. Lin, C. H. Yip, *The EKR-module property of pseudo-Paley graphs of square order*, arXiv:2201.03100

On Peisert-type graphs with strict-EKR property (I)

Corollary 3.6 ([AGLY22, Corollary 1.8])

If $q > (m-1)^2$, then all Peisert-type graphs of type (m, q) satisfy the strict-EKR property. In particular, if $d > \frac{q+1}{\sqrt{q}+1}$ and $d \mid (q+1)$, then the d-Paley graph $GP(q^2, d)$ has the strict-EKR property.

[AGLY22] S. Asgarli, S. Goryainov, H. Lin, C. H. Yip, *The EKR-module property of pseudo-Paley graphs of square order*, arXiv:2201.03100

On Peisert-type graphs with strict-EKR property (II)

It is natural to examine when a Peisert-type graph X enjoys the strict-EKR property. While we do not have a general answer to this problem, we exhibit an infinite family of Peisert-type graphs which fail to satisfy the strict-EKR property. The following theorem shows that the condition $q > (m-1)^2$ in Corollary 3.6 is sharp when q is a square.

Theorem 3.7 ([AGLY22, Theorem 1.9])

Let q be an odd prime power which is not a prime. Then there exists a Peisert-type graph X of order q^2 such that X fails to have the strict-EKR property. In particular, if q is a square, then there exists a Peisert-type graph X of type $(\sqrt{q}+1, q)$ which fails to have the strict-EKR property.

[AGLY22] S. Asgarli, S. Goryainov, H. Lin, C. H. Yip, *The EKR-module property of pseudo-Paley graphs of square order*, arXiv:2201.03100

Chromatic number and ERK theorem

The chromatic number of a graph X, denoted $\chi(X)$, is the smallest number of colours needed to colour the vertices of X so that no two adjacent vertices share the same colour.

We remark that one can prove the original EKR theorem using the (fractional) chromatic number of Kneser graphs [GR01, Theorem 7.8.1].

It is known that the chromatic number is lower bounded by the clique number, that is, $\omega(X) \leq \chi(X)$.

[GR01] C. Godsil and G. Royle, *Algebraic graph theory*, Graduate Texts in Mathematics, vol. 207, Springer-Verlag, New York, 2001.

On chromatic and clique numbers of generalised Paley graphs

Broere, Döman, and Ridley [BDR88] showed that if $d > 1$ and $d \mid (q+1)$, then both the chromatic number and the clique number of the generalised Paley graph $GP(q^2, d)$ is q.

The converse of this result was proved by Schneider and Silva [SS15, Theorem 4.7].

A stronger converse was proved recently in [Y21].

[BDR88] I. Broere, D. Dőman, and J. N. Ridley, *The clique numbers and chromatic numbers of certain Paley graphs*, Quaestiones Math. 11 (1988), no. 1, 91–93.

[SS15] C. Schneider and A. C. Silva, *Cliques and colorings in generalised Paley graphs and an approach to synchronization*, J. Algebra Appl. 14 (2015), no. 6, 1550088, 13.

[Y21] C. H. Yip, Gauss sums and the maximum cliques in generalised Paley graphs of square order, Funct. Approx. Comment. Math. (2021)

Chromatic and clique numbers of Peisert-type graphs graphs

Our following theorem computes both the chromatic and the clique number of all Peisert-type graphs, hence extending the first result on generalised Paley graphs since $GP(q^2, d)$ with $d \mid (q+1)$ is a Peisert-type graph.

Theorem 3.8 ([AGLY22, Theorem 1.7])

Let X be a Peisert-type graph of order q^2. Then $\omega(X) = \chi(X) = q$.

[AGLY22] S. Asgarli, S. Goryainov, H. Lin, C. H. Yip, *The EKR-module property of pseudo-Paley graphs of square order*, arXiv:2201.03100

Open problems (IV)

Let X be a Peisert-type graph, and W be the vector space generated by the characteristic vectors of the canonical cliques in X. As we mentioned above, there is no obvious choice of a non-trivial group action on W. Finding such a group action, already in the case of the Paley graph, may give new insights on the EKR theorems.

Problem 3.1

Does there exist a 2-transitive permutation group G that acts linearly on the vector space W generated by the characteristic vectors of canonical cliques in the Paley graph P_{q^2}?

Another problem, motivated by the counterexamples found in Theorem 3.7, is the following.

Problem 3.2

Characterise Peisert-type graphs with the strict-EKR property.

Open problems (V)

Peisert-type graphs of order q^2 can be analogously defined in the case when q is a power of 2.

Problem 3.3

Investigate EKR properties of Peisert-type graphs in characteristic 2.

Note that we already have some progress on Problem 3.3.

4. Tightness of the weight-distribution bound for some strongly regular graphs

Eigenfunctions of graphs

Let $\Gamma = (V, E)$ be a k-regular graph on n vertices and λ be an eigenvalue of its adjacency matrix A. Let $u = (u_1, \ldots, u_n)^t$ be an eigenvector of A corresponding to λ. Then u defines a function $f_u : V \mapsto \mathbb{R}$, which is called a λ-eigenfunction of Γ.

For an eigenfunction f_u of Γ, the *support* is the set

$$Supp(f_u) := \{x \in V \mid f_u(x) \neq 0\}.$$

MS-problem

The following problem was first formulated in [VK15] (see also [SV21] for the motivation and details).

Problem 4.1 (MS-problem)

Given a graph Γ and its eigenvalue λ, find the minimum cardinality of the support of a λ-eigenfunction of Γ.

A λ-eigenfunction having the minimum cardinality of support is called optimal.

Problem 4.2 (Strong MS-problem)

Given a graph Γ and its eigenvalue λ, characterise optimal λ-eigenfunctions of Γ.

[VK15] K. V. Vorobev, D. S. Krotov, *Bounds for the size of a minimal 1-perfect bitrade in a Hamming graph*, Journal of Applied and Industrial Mathematics 9(1) (2015) 141–146.

[SV21] E. Sotnikova, A. Valyuzhenich, *Minimum supports of eigenfunctions of graphs: a survey*, Art Discrete Appl. Math. 4 (2021), no. 2, Paper No. 2.09, 34 pp.

A survey on Problem 4.2

Recently, Problem 4.2 was solved for several classes of graphs:

- all eigenvalues of Hamming graphs $H(n, q)$ when $q = 2$ or $q > 4$ and some eigenvalues of $H(n, q)$ when $q = 3, 4$;
- all eigenvalues of Johnson graphs (asymptotically);
- the smallest eigenvalue of Hamming, Johnson and Grassmann graphs;
- the largest non-principal eigenvalue of a Star graph S_n, $n \geq 8$;
- the largest non-principal eigenvalue of Doob graphs.

A survey on Problem 4.1

Excepting the results from the previous slide, Problem 4.1 was solved for several more classes of graphs:

- both non-principal eigenvalues of Paley graphs of square order;
- strongly regular bilinear forms graphs over a prime field.

Weight-distribution bound

Let Γ be a distance-regular graph of diameter $D(\Gamma)$ with intersection array $(b_0, b_1, \ldots, b_{D(\Gamma)-1}; c_1, c_2, \ldots, c_{D(\Gamma)})$ and nonprincipal eigenvalue λ.

Theorem 4.1 (Weight-distribution bound, [KMP16, Corollary 1])

A λ-eigenfunction f of Γ has at least $\sum_{i=0}^{D(G)} |W_i|$ nonzeros, where

$$W_0 = 1,$$

$$W_1 = \lambda$$

and

$$W_i = \frac{(\lambda - a_{i-1})W_{i-1} - b_{i-2}W_{i-2}}{c_i}.$$

[KMP16] D. S. Krotov, I. Yu. Mogilnykh, V. N. Potapov, *To the theory of q-ary Steiner and other-type trades*, Discrete Mathematics 339 (3) (2016) 1150–1157.

Known results when the weight-distribution bound is tight

- the eigenvalue -1 of the Boolean Hamming graph of an odd dimension and the minimum eigenvalue of an arbitrary Hamming graph;
- both non-principal eigenvalues of Paley graphs of square order;
- the minimum eigenvalue of Johnson graphs;
- the minimum eigenvalue of Grassmann graphs;
- the minimum eigenvalue of strongly regular bilinear forms graphs over a prime field.

Tightness of the weight-distribution bound for the smallest eigenvalue of a DRG

It was shown in [KMP16] that, for the smallest eigenvalue of a distance-regular graph Γ, the tightness of the weight-distribution bound is equivalent to the existence of an isometric bipartite distance-regular induced subgraph $T_0 \cup T_1$, where T_0 and T_1 are parts, such that an optimal eigenfunction, up to multiplication by a non-zero constant, has the following form:

$$f(x) = \begin{cases} 1, & \text{if } x \in T_0; \\ -1, & \text{if } x \in T_1; \\ 0, & \text{otherwise.} \end{cases}$$

[KMP16] D. S. Krotov, I. Yu. Mogilnykh, V. N. Potapov, *To the theory of q-ary Steiner and other-type trades*, Discrete Mathematics 339 (3) (2016) 1150–1157.

Tightness of the weight-distribution bound for a non-principal eigenvalue of an SRG

If Γ is a strongly regular graph with non-principal eigenvalues θ, τ, where $\tau < 0 < \theta$, the following holds.

Theorem 4.2 ([KMP16], Weight-distribution bound for SRG)

(1) An τ-eigenfunction f of Γ has at least (-2τ) nonzeros; $|Supp(f)|$ meets the bound if and only if there exists an induced complete bipartite subgraph with parts T_0, T_1 of size $-\tau$;
(2) An θ-eigenfunction f of Γ has at least $2(\theta+1)$ nonzeros; $|Supp(f)|$ meets the bound if and only if there exists an induced disjoint union of two cliques T_0, T_1 of size $\theta+1$.

[KMP16] D. S. Krotov, I. Yu. Mogilnykh, V. N. Potapov, *To the theory of q-ary Steiner and other-type trades*, Discrete Mathematics 339 (3) (2016) 1150–1157.

Tightness of the weight-distribution bound for Paley graphs of square order

In [GKSV18], for Paley graphs $P(q^2)$, we showed the tightness of the weight-distribution bound for both non-principal eigenvalues, which are $\tau = \frac{-1-q}{2}$ and $\theta = \frac{-1+q}{2}$.

Let β be a primitive element in $\mathbb{F}_{q^2}$. Put $\omega := \beta^{q-1}$. Then $Q = \langle \omega \rangle$ is the subgroup of order $q+1$ in $\mathbb{F}_{q^2}^*$.

Facts about Q:

- Q is an oval in the corresponding affine plane;
- Q is the kernel of the norm mapping $N : \mathbb{F}_{q^2}^* \mapsto \mathbb{F}_q^*$, which means that $Q = \{\gamma \in \mathbb{F}_{q^2}^* \mid \gamma^{q+1} = 1\}$, or, equivalently, $Q = \{x + y\alpha \mid x, y \in \mathbb{F}_q, x^2 - y^2 d = 1\}$, where d is a non-square in $\mathbb{F}_q^*$ and $\alpha^2 = d$.

[GKSV18] S. Goryainov, V. Kabanov, L. Shalaginov, A. Valyuzhenich, *On eigenfunctions and maximal cliques of Paley graphs of square order*, Finite Fields and Their Applications 52 (2018) 361–369.

Tightness of the weight-distribution bound for Paley graphs of square order

Let $Q_0 = \langle \omega^2 \rangle$ and $Q_1 = \omega Q_0$.

Facts about Q:

- if $q \equiv 1(4)$, then $Q = Q_0 \cup Q_1$ induces a complete bipartite graph with parts Q_0 and Q_1;
- if $q \equiv 3(4)$, then $Q = Q_0 \cup Q_1$ induces a pair of disjoint cliques Q_0 and Q_1.

Corollary 4.3, [GKSV18, Theorem 2]

The weight-distribution bound is tight for both non-principal eigenvalues of Paley graphs of square order.

Knowing the structure of Q, we were also able to construct new maximal cliques of the second largest known size in Paley graphs of square order (see [GKSV18]).

[GKSV18] S. Goryainov, V. Kabanov, L. Shalaginov, A. Valyuzhenich, *On eigenfunctions and maximal cliques of Paley graphs of square order*, Finite Fields and Their Applications 52 (2018) 361–369.

Generalised Paley graphs of square order; WDB for the smallest eigenvalue

Let $m > 1$ be a positive integer. Let q be an odd prime power, $q \equiv 1\ (2m)$. The m-Paley graph on $\mathbb{F}_q$, denoted $GP(q, m)$, is the Cayley graph $Cay(\mathbb{F}_q^+, (\mathbb{F}_q^*)^m)$, where $(\mathbb{F}_q^*)^m$ is the set of m-th powers in $\mathbb{F}_q^*$.

We consider the graphs $GP(q^2, m)$, where q is an odd prime power and m divides $q + 1$; these graphs are strongly regular and form a generalisation of Paley graphs of square order (the usual Paley graphs of square order are just 2-Paley graphs of square order).

The eigenvalues of $GP(q^2, m)$ are $\tau = (-\frac{q+1}{m})$ and $\theta = \frac{(m-1)q-1}{m}$.

Given an odd prime power q and an integer $m > 1$ such that m divides $q + 1$, a $(-\frac{q+1}{m})$-eigenfunction of the generalised Paley graph $GP(q^2, m)$ has at least $\frac{2(q+1)}{m}$ non-zeroes.

Structure of Q (I)

Let us divide Q into m parts

$$Q = Q_0 \cup Q_1 \cup \ldots \cup Q_{m-1},$$

where $Q_0 = \langle \omega^m \rangle$, $Q_1 = \omega Q_0$, $\ldots$, $Q_{m-1} = \omega^{m-1} Q_0$.

Proposition 4.4 ([GSY22, Lemma 3.8])

Let q be a prime power and m be an integer such that $m > 1$, m divides $q+1$. The mapping $\gamma \mapsto \gamma^{q-1}$ is a homomorphism from $\mathbb{F}_{q^2}^*$ to Q. Moreover, an element γ is an m-th power in $\mathbb{F}_{q^2}^*$ if and only if γ^{q-1} is an m-th power in Q.

[GSY22] S. Goryainov, L. Shalaginov, C. H. Yip, *On eigenfunctions and maximal cliques of generalised Paley graphs of square order*, arXiv:2203.16081

Structure of Q (II)

Let us divide Q into m parts

$$Q = Q_0 \cup Q_1 \cup \ldots \cup Q_{m-1},$$

where $Q_0 = \langle \omega^m \rangle$, $Q_1 = \omega Q_0$, $\ldots$, $Q_{m-1} = \omega^{m-1} Q_0$.

Proposition 4.5 ([GSY22, Lemma 3.10])

Let γ be an arbitrary element from Q, $\gamma \neq 1$. Then, for the image of $(\gamma - 1)$ under the action of the homomorphism, the following equality holds:

$$(\gamma - 1)^{q-1} = -\frac{1}{\gamma}.$$

[GSY22] S. Goryainov, L. Shalaginov, C. H. Yip, *On eigenfunctions and maximal cliques of generalised Paley graphs of square order*, arXiv:2203.16081

Structure of Q (III)

The following theorem basically states that each of the sets $Q_0, Q_1 \ldots, Q_{m-1}$ induces either a clique or an independent set, and there are at most two cliques among them.

Moreover, the theorem states that for every independent set Q_{i_1}, there exists uniquely determined independent set Q_{i_2} among $Q_0, Q_1 \ldots, Q_{m-1}$ such that there are all possible edges between Q_{i_1} and Q_{i_2} and there are no edges between Q_{i_1} and $Q \setminus Q_{i_2}$.

Structure of Q (IV)

Theorem 4.6 ([GSY22, Theorem 4.1])

Given an odd prime power q and an integer $m > 1$, m divides $q+1$, the following statements hold for the subgraph of $GP(q^2, m)$ induced by Q.
(1) If m divides $\frac{q+1}{2}$ and m is odd, then Q_0 is a clique, and $Q_1, \ldots, Q_{m-1}$ are independent sets; moreover, for any distinct i_1, i_2 such that $0 \leq i_1 < i_2 \leq m-1$, there are all possible edges between the sets Q_{i_1} and Q_{i_2} if $i_1 + i_2 \equiv 0 \ (mod\ m)$, and there are no such edges if $i_1 + i_2 \not\equiv 0 \ (mod\ m)$. In particular, $Q_1 \cup Q_{m-1}, \ldots, Q_{\frac{m-1}{2}} \cup Q_{\frac{m+1}{2}}$ induce $\frac{m-1}{2}$ complete bipartite graphs.

[GSY22] S. Goryainov, L. Shalaginov, C. H. Yip, *On eigenfunctions and maximal cliques of generalised Paley graphs of square order*, arXiv:2203.16081

Structure of Q (V)

(2) If m divides $\frac{q+1}{2}$ and m is even, then $Q_0, Q_{\frac{m}{2}}$ are cliques, and $Q_1, \ldots, Q_{\frac{m}{2}-1}, Q_{\frac{m}{2}+1}, \ldots, Q_{m-1}$ are independent sets; moreover, for any distinct i_1, i_2 such that $0 \leq i_1 < i_2 \leq m-1$, there are all possible edges between the sets Q_{i_1} and Q_{i_2} if

$$i_1 + i_2 \equiv 0 \ (mod\ m) \text{ and } \{i_1, i_2\} \neq \{0, \frac{m}{2}\}$$

and there are no such edges if

$$i_1 + i_2 \not\equiv 0 \ (mod\ m) \text{ or } \{i_1, i_2\} = \{0, \frac{m}{2}\}.$$

In particular, $Q_1 \cup Q_{m-1}, \ldots, Q_{\frac{m}{2}-1} \cup Q_{\frac{m}{2}+1}$ induce $(\frac{m}{2} - 1)$ complete bipartite graphs.

Structure of Q (VI)

(3) If m does not divide $\frac{q+1}{2}$, then m is even.
(3.1) If $\frac{m}{2}$ is odd, then $Q_0, Q_1, \ldots, Q_{m-1}$ are independent sets; moreover, for any distinct i_1, i_2 such that $0 \leq i_1 < i_2 \leq m-1$, there are all possible edges between the sets Q_{i_1} and Q_{i_2} if

$$i_1 + i_2 \equiv \frac{m}{2} \ (mod\ m),$$

and there are no such edges if

$$i_1 + i_2 \not\equiv \frac{m}{2} \ (mod\ m).$$

In particular, if $m = 2$, $Q = Q_0 \cup Q_1$ is a complete bipartite graph; if $m \geq 6$,
$Q_0 \cup Q_{\frac{m}{2}}, \ldots, Q_{\frac{m-2}{4}} \cup Q_{\frac{m+2}{4}}, Q_{\frac{m}{2}+1} \cup Q_{m-1}, \ldots, Q_{\frac{3m-2}{4}} \cup Q_{\frac{3m+2}{4}}$
induce $\frac{m}{2}$ complete bipartite graphs.

Structure of Q (VII)

(3.2) If $\frac{m}{2}$ is even, then $Q_{\frac{m}{4}}, Q_{\frac{3m}{4}}$ are cliques, and $Q_0, \ldots, Q_{\frac{m}{4}-1}, Q_{\frac{m}{4}+1}, \ldots, Q_{\frac{3m}{4}-1}, Q_{\frac{3m}{4}+1}, \ldots, Q_{m-1}$ are independent sets; moreover, for any distinct i_1, i_2 such that $0 \leq i_1 < i_2 \leq m-1$, there are all possible edges between the sets Q_{i_1} and Q_{i_2} if

$$i_1 + i_2 \equiv \frac{m}{2} \ (mod\ m) \text{ and } \{i_1, i_2\} \neq \{\frac{m}{2}, \frac{3m}{2}\},$$

and there are no such edges if

$$i_1 + i_2 \not\equiv \frac{m}{2} \ (mod\ m) \text{ or } \{i_1, i_2\} = \{\frac{m}{2}, \frac{3m}{2}\}.$$

In particular, if $m = 4$, $Q_0 \cup Q_2$ is a complete bipartite graph; if $m \geq 8$, then
$Q_0 \cup Q_{\frac{m}{2}}, \ldots, Q_{\frac{m-4}{4}} \cup Q_{\frac{m+4}{4}}, Q_{\frac{m}{2}+1} \cup Q_{m-1}, \ldots, Q_{\frac{3m-4}{4}} \cup Q_{\frac{3m+4}{4}}$
induce $\frac{m-2}{2}$ complete bipartite graphs.

Structure of Q (VIII) and tightness of WDB for the smallest eigenvalue of $GP(q^2, m)$

Corollary 4.7 ([AGY22, Corollary 4.2])

Let q be an odd prime power and m be an integer $m \geq 2$, m divides $q+1$. Then, except for the case $m = 2$ and 2 divides $\frac{q+1}{2}$, there is at least one pair Q_{i_1}, Q_{i_2} among $Q_0, \ldots, Q_{m-1}$ such that $Q_{i_1} \cup Q_{i_2}$ induces a complete bipartite subgraph.

Corollary 4.8 ([AGY22, Theorem 4.3])

Let q be an odd prime power and m be an integer $m \geq 2$, m divides $q+1$. Then the weight-distribution bound is tight for the eigenvalue $(-\frac{q+1}{m})$ of $GP(q^2, m)$.

[GSY22] S. Goryainov, L. Shalaginov, C. H. Yip, *On eigenfunctions and maximal cliques of generalised Paley graphs of square order*, arXiv:2203.16081

Open problems (VI)

Problem 4.1

Investigate the maximality or near-maximality of the cliques from Theorem 4.6.

Thank you for your attention!